Modeling and Learning for Dynamical Systems

Lecture 5

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Nonparametric System Identification

Motivation (Repeated)

Physical modeling might not be enough to obtain a useful model:

- Some model parameters might be unknown
- Some physical mechanisms might be unknown
- The system might be too complex (to finish the modeling in reasonable time and with limited resources)

One convenient alternative: **System identification** (data-driven modeling)

- Key idea: Measure the input and output signals and fit a model to data such that it explains the input-output relation well
- Key aspect: The signals are measured at discrete time instances (sampling) $\Rightarrow$ Natural to use discrete-time models

Nonparametric Models

A *parameterized* model:

$$y(t) = G(q, \theta)u(t) + H(q, \theta)e(t),$$

is defined using a parameter vector θ with fixed dimension.

A *nonparametric* model gives a description of a system without (explicitly) defining a parameter vector of fixed (finite) dimension.

- Time-domain and frequency-domain approaches
- Usually open-loop data
- The model complexity often depends on the amount of data used

Nonparametric Time-domain Methods

Four time-domain methods:

- Impulse-response analysis (measure the impulse response)
- Step-response analysis (measure the step response)
- Correlation analysis
- Estimate an FIR model using the least-squares method

Impulse-Response Analysis

Model of general discrete-time system:

$$y(t) = \sum_{k=0}^{\infty} g_k u(t-k) + v(t)$$

Impulse-response analysis:

- Use an impulse as input and measure the output $y(t) = g_t + v(t)$
- Result is easy to interpret
- Increasing the impulse amplitude or averaging several impulse responses improve the accuracy
- Time-consuming way to get accurate estimate
- Impulses, especially large ones, can be hard to use in some applications

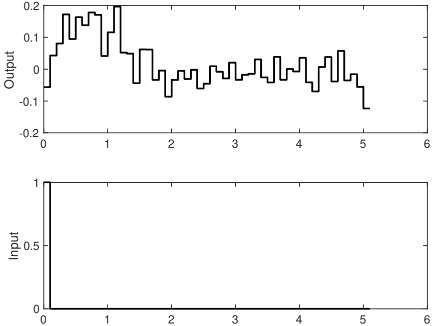
Example: Impulse Experiment

True system:

$$G(s) = \frac{5}{s^2 + 2s + 5}$$

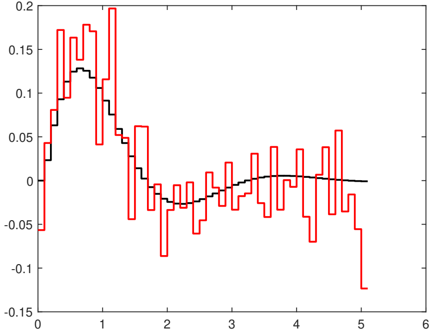
- Sampling time: $T = 0.1\,\text{s}$
- Measurement noise: white noise $v(t) \in \mathcal{N}(0, 0.05^2)$
- $u =$ impulse

Input-output data:



Example: Impulse Experiment...

Result (estimate in red, true impulse response in black):

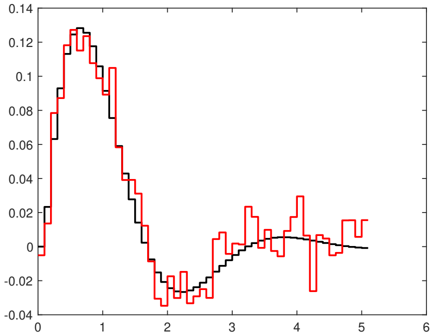


The result of one impulse experiment is a rather noisy estimate of the true impulse response

Example: Impulse Experiment...

Taking the average over 20 impulse experiments improves the estimate, but the effects of noise are still clearly visible

Result (estimate in red, true impulse response in black):



Correlation Analysis

- Remove the means from u and y
- Estimate a model of the input $u(t) = \frac{1}{A(q)} e(t)$, where $e(t)$ is assumed to be white noise, and prefilter the input and output with $\hat{A}(q)$:

$$u_F(t) = \hat{A}(q)u(t)$$
$$y_F(t) = \hat{A}(q)y(t)$$

- Compute the estimates

$$\hat{R}^N_{y_F u_F}(\tau) = \frac{1}{N - \tau} \sum_{t=\tau+1}^{N} y_F(t) u_F(t - \tau)$$
$$\hat{R}^N_{u_F}(0) = \frac{1}{N} \sum_{t=1}^{N} u_F(t)^2$$

- Finally, compute the impulse response estimate

$$\hat{g}_\tau = \frac{\hat{R}^N_{y_F u_F}(\tau)}{\hat{R}^N_{u_F}(0)}$$

Correlation Analysis...

- The method does not require a particular input signal
- A larger input or longer experiment improves the accuracy
- No truncation errors if only a part of the impulse response is estimated
- The order of $A(q)$ has to be selected (usually not an issue)
- Matlab command: `cra`

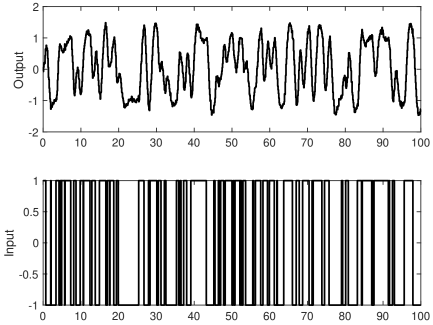
Example: Correlation Analysis

True system:

$$G(s) = \frac{5}{s^2 + 2s + 5}$$

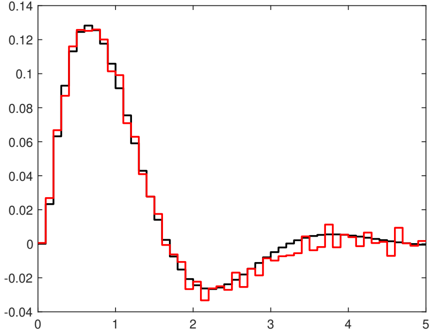
- Sampling time: $T = 0.1\,\mathrm{s}$
- Measurement noise: white noise $v(t) \in \mathcal{N}(0, 0.05^2)$
- $u =$ telegraph signal (switching from -1 to 1 with a particular probability)
- Data length: $N = 1000$

Input-output data:



Example: Correlation Analysis. . .

Result (estimate in red, true impulse response in black):

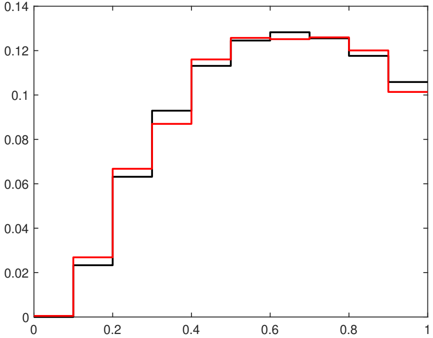


Perform correlation analysis with $\hat{A}(q)$ of order 50. Here, 50 components of the impulse response have been computed.

Example: Correlation Analysis. . .

Computing only 10 components of the impulse response does not change the accuracy.

Result (estimate in red, true impulse response in black):



Least-Squares Estimate of FIR Model

Estimate a finite impulse response (FIR) by minimizing

$$\sum_{t=1}^{N} (y(t) - \theta^T \varphi(t))^2$$

with respect to θ. Here:

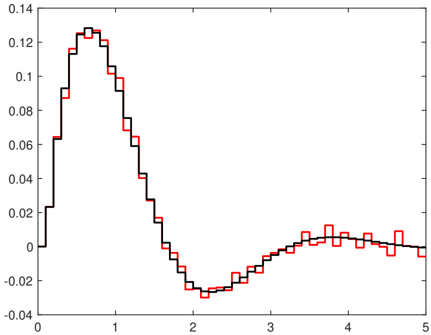
$$\begin{aligned} \theta^T &= \begin{pmatrix} g_0 & g_1 & \dots & g_n \end{pmatrix} \\ \varphi(t)^T &= \begin{pmatrix} u(t) & u(t-1) & \dots & u(t-n) \end{pmatrix} \end{aligned}$$

- The method does not require a particular input signal
- A larger input or longer experiment improves the accuracy
- Truncation errors if only a part of the impulse response is estimated
- Matlab command: `arx`

Example: FIR Model Estimation

Estimate an FIR model with 50 components using the least-squares method.

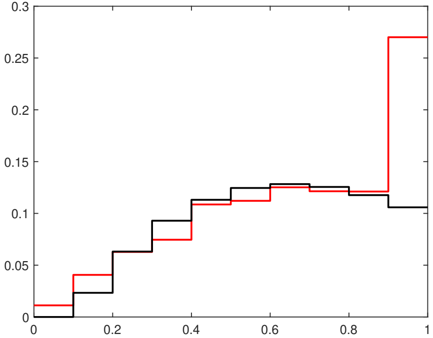
Result (estimate in red, true impulse response in black):



Example: FIR Model Estimation

Reducing the length of the estimated impulse response results in truncation errors.

Result (estimate in red, true impulse response in black):



Nonparametric Frequency-domain Methods

Three frequency-domain methods:

- Frequency analysis (sine-wave testing, directly or using a correlation approach)
- Fourier analysis
- Spectral analysis

Periodograms

Consider a signal $w(t)$. The **periodogram** of the signal is another name for the discrete-time Fourier transform of the signal:

$$W_N(e^{i\omega}) = \sum_{k=1}^{N} w(k)e^{-i\omega k}$$

- Periodic signals give clear peaks in the periodogram
- In general, the periodogram is an unbiased but noisy estimate of the frequency contents of the signal
- The variance can be reduced by averaging periodograms computed from different datasets (Welch's method)
- The periodogram can be smoothed by using a window function (Blackman-Tukey's method)

Blackman-Tukey's Method

Estimate spectral densities using

$$\hat{\Phi}_u^N(\omega) = \sum_{k=-\gamma}^{\gamma} w_\gamma(k)\hat{R}_u^N(k)e^{-i\omega k}$$

$$\hat{R}_u^N(\tau) = \frac{1}{N-\tau}\sum_{t=\tau+1}^{N} u(t)u(t-\tau)$$

where w_γ is a **window function** of size γ

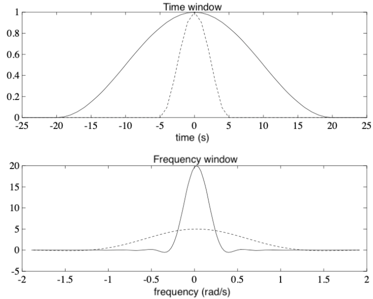
- A large $\gamma \Rightarrow$ wide time window, high frequency resolution (less averaging)
- A small $\gamma \Rightarrow$ narrow time window, low frequency resolution (more averaging)
- Some classical windows: Hamming, Bartlett, Parzen

Hamming Window

The Hamming window:

$$w_\gamma(k) = \begin{cases} \frac{1}{2}(1+\cos(\frac{\pi k}{\gamma})), & |k| < \gamma \\ 0 & |k| \geq \gamma \end{cases}$$

Solid line: $\gamma = 20$, dashed line $\gamma = 5$



Cross-Spectral Densities

Estimates of cross-spectral densities can also be computed using window functions:

$$\hat{\Phi}_{yu}^{N}(\omega) = \sum_{k=-\gamma}^{\gamma} w_{\gamma}(k)\hat{R}_{yu}^{N}(k)e^{-i\omega k}$$

$$\hat{R}_{yu}^{N}(\tau) = \frac{1}{N-\tau}\sum_{t=\tau+1}^{N} y(t)u(t-\tau)$$

Fourier Analysis

The **empirical transfer function estimate (ETFE)**:

$$\hat{\hat{G}}_N(e^{i\omega}) = \frac{Y_N(e^{i\omega})}{U_N(e^{i\omega})}$$

$Y_N(e^{i\omega})$ and $U_N(e^{i\omega})$ (assumed $\neq 0$) are the periodograms of the output and input signals, respectively.

- Gives a noisy but unbiased estimator of the frequency response (often computed at the DFT frequencies $\omega = \frac{2\pi k}{N}$) for general input signals (and the variance does not decay with the data length)
- Gives a very useful estimator of the frequency response at the excited frequencies for a periodic input (and the variance decays with the data length)
- Can be smoothed using frequency domain windows (gives a bias-variance tradeoff)
- Matlab command: `etfe`

Spectral Analysis

The **spectral analysis estimate (SPA)**:

$$\hat{G}_N(e^{i\omega}) = \frac{\hat{\Phi}_{yu}^N(\omega)}{\hat{\Phi}_u^N(\omega)}$$

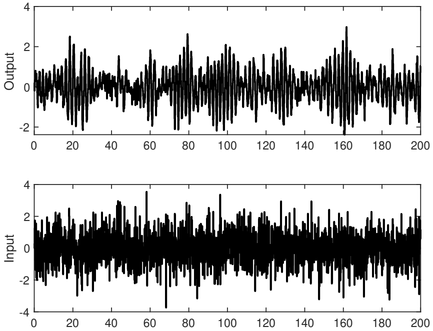
- The choice of window function gives a bias-variance tradeoff
- Matlab command: `spa`

Example: Resonant System (ETFE and SPA)

Consider a resonant system

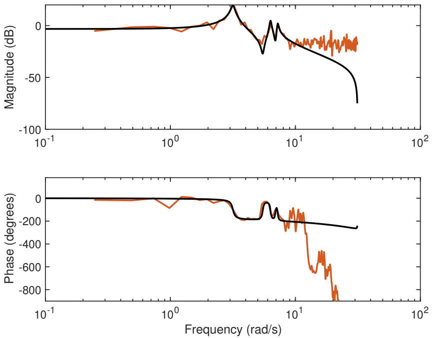
- Sampling time: $T = 0.1\,\mathrm{s}$
- The standard deviation of the measurement noise is 20% of the standard deviation of the noise-free output
- $u =$ white noise with distribution $\mathcal{N}(0, 1^2)$
- Data length: $N = 2000$

Input-output data:

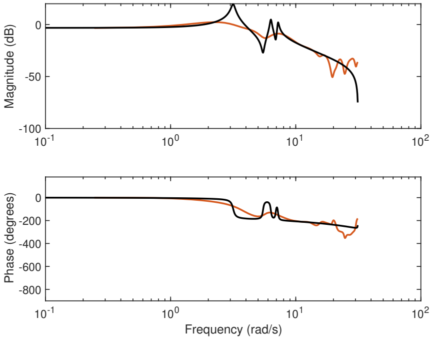


Example: Resonant System (ETFE and SPA)...

ETFE estimate with default setting (no windowing)



SPA estimate with default setting $\gamma = 30$

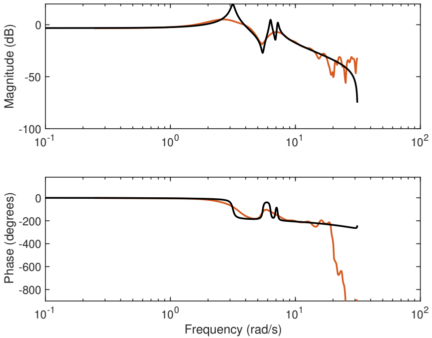


(estimates in red, true frequency response in black)

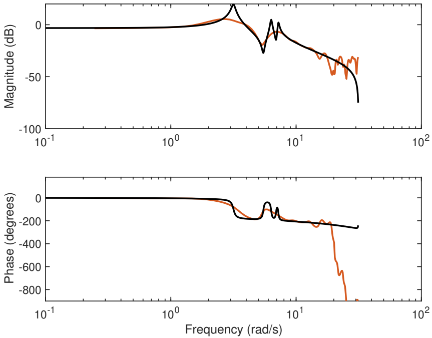
The two estimates are quite different. In practice (without black lines), it is not so easy to say which resonances there are in the true system.

Example: Resonant System (ETFE and SPA)...

ETFE estimate with $\gamma = 50$



SPA estimate with $\gamma = 50$

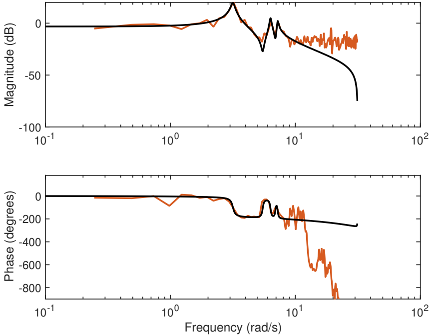


(estimates in red, true frequency response in black)

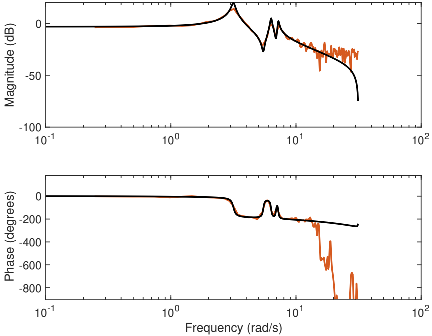
The two smoothed estimates are quite similar. Better to compare the smoothed SPA estimate with the unsmoothed ETFE.

Example: Resonant System (ETFE and SPA)...

ETFE estimate with default setting (no windowing)



SPA estimate with $\gamma = 200$



(estimates in red, true frequency response in black)

The SPA estimate is in line with the default ETFE estimate but less noisy. The resonances can be seen quite clearly.

The Effect of Feedback

- The methods discussed here all depend on an assumption that the measurement noise is uncorrelated with the input
- This is not the case if feedback is present
- These methods can give wrong (biased) results if data is collected in closed loop (with feedback)

Summary

Nonparametric System Identification

- Impulse response estimates via correlation analysis
- Finite impulse response (FIR) estimates via the least-squares method
- Frequency response estimates via Fourier or spectral analysis
- Window functions

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