# Modeling and Learning for Dynamical Systems

Lecture 4

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# Sampling and Discrete-Time Models

# Motivation

Physical modeling might not be enough to obtain a useful model:

- Some model parameters might be unknown
- Some physical mechanisms might be unknown
- The system might be too complex (to finish the modeling in reasonable time and with limited resources)

One convenient alternative: **System identification** (data-driven modeling)

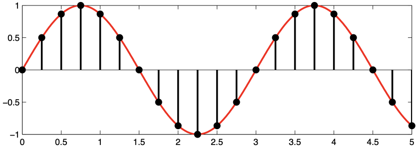
- Key idea: Measure the input and output signals and fit a model to data such that it explains the input-output relation well
- Key aspect: The signals are measured at discrete time instances (sampling) $\Rightarrow$ Natural to use discrete-time models

# Sampling

**Sampling:** Information about a signal is collected at discrete time instances $t_k$ (the sampling times)

Uniform sampling: $t_k = kT$

- Sampling time: $T$
- Sampling frequency: $f_s = \frac{1}{T}$
- Sampling angular frequency: $\omega_s = \frac{2\pi}{T}$



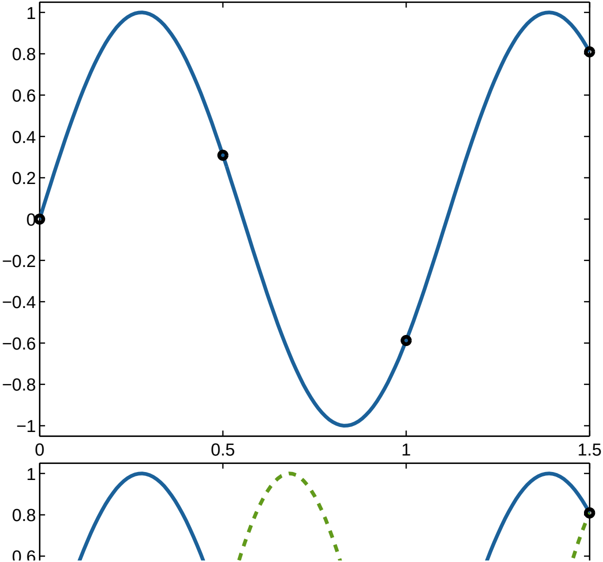

Continuous time:

$$y(t) = \sin(\omega t)$$

Discrete time:

$$y(t_k) = \sin(\omega t_k)$$

# Aliasing



# Sampling Theory

How should the sampling frequency be chosen to avoid aliasing?

**Sampling Theorem:** A signal that does not contain any signal components above the angular frequency $\omega_0$ can be reconstructed exactly from sampled data if the sampling frequency $\omega_s$ satisfy the inequality $\omega_0 \leq \frac{\omega_s}{2}$.

The frequency $\omega_N = \frac{\omega_s}{2}$ is known as **the Nyquist frequency**.

# Sampling Theory...

Poisson's summation formula gives a relation between the Fourier transform $W(i\omega)$ of a continuous-time signal $w(t)$ and the discrete Fourier transform $W^{(T)}(e^{i\omega T})$ of the sampled signal $w(kT)$:

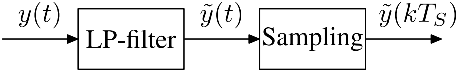
$$W^{(T)}(e^{i\omega T}) = \sum_{r=-\infty}^{\infty} W(i(\omega + r\omega_s))$$

Observations:

- $W^{(T)}(e^{i\omega T})$ is periodic with period $\omega_s$
- It is enough to consider the interval $-\omega_N \leq \omega \leq \omega_N$
- Frequency components in $W(i\omega)$ outside this interval are misinterpreted (aliasing)

# Sampling in Practice

In practice, there could always be signal components above the Nyquist frequency (e.g., measurement noise) that could result in aliasing. Hence, it is standard practice to low-pass filter the continuous-time signal (with an anti-aliasing filter) before the sampling.




(This filter eliminates signal components above $\omega_N$ but leaves slow signal components unchanged.)

# State-Space Models in Discrete Time

A discrete-time state-space model:

$$
\begin{aligned}
x(t_{k+1}) &= f(x(t_k), u(t_k)), \quad k = 0, 1, 2, \dots \\
y(t_k) &= h(x(t_k), u(t_k))
\end{aligned}
$$

where $x$ are states, $u$ inputs and $y$ outputs (all vectors)

# Sampling of Linear State-Space Model

Sampling of a model means that a discrete-time model is obtained from a continuous-time one.

Linear continuous-time state-space model:

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \end{cases} \qquad (\Leftrightarrow \;\; G(s) = C(sI - A)^{-1}B)$$

Assume that the input is **piecewise constant**, i.e. $u(t) = u(kT)$, $kT \leq t < kT + T$. Solution:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

# Sampling of Linear State-Space Model...

**Result:**

$$\begin{cases} x(kT+T) & = Fx(kT) + Gu(kT) \\ y(kT) & = Hx(kT) \end{cases}$$

where

$$F = e^{AT}, \quad G = \int_0^T e^{A\sigma} B \, d\sigma \quad \text{and} \quad H = C$$

(This is an exact discrete-time description of the continuous-time model at the sampling times.)

$e^{AT}$ can be calculated as

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$

N.B. Other approaches (e.g., the standard Euler method) can be used if the model is nonlinear or the input is not zero-order hold

# Discrete-Time to Continuous-Time?

**Reversed problem:** Find $A$ such that $e^{AT} = F$. Then, $B$ can be calculated as

$$B = \left( \int_0^T e^{A\sigma} \, d\sigma \right)^{-1} G$$

Complications:

- $e^{AT} = F$ might lack solution
- $e^{AT} = F$ might have several solutions

# Solution

A discrete-time state-space model

$$\begin{aligned} x(kT+T) &= Fx(kT) + Gu(kT) \\ y(kT) &= Hx(kT) \end{aligned}$$

is its own solution algorithm:

$$\begin{aligned} x(kT) &= Fx(kT-T) + Gu(kT-T) \\ &= F(Fx(kT-2T) + Gu(kT-2T)) + Gu(kT-T) \\ &= \dots \\ &= F^{(k-k_0)}x(k_0T) + \sum_{l=k_0}^{k-1} F^{(k-l-1)}Gu(lT) \end{aligned}$$

# Z Transform

The discrete-time counterpart to the Laplace transform is called the **z transform**:

$$Y(z) = \mathcal{Z}\{y(kT)\}(z) = \sum_{k=0}^{\infty} y(kT) z^{-k}$$

$(y(kT) = 0$ for $k < 0)$

# Z Transform...

Some properties:

$$
\begin{aligned}
\mathcal{Z}\{ay(kT) + bv(kT)\} &= aY(z) + bV(z) \\
\mathcal{Z}\{y(kT - T)\} &= z^{-1}Y(z) + y(-T) \\
\mathcal{Z}\{y(kT + T)\} &= zY(z) - zy(0) \\
\mathcal{Z}\{\sum_{m=0}^{k} y(kT - mT)v(mT)\} &= Y(z)V(z)
\end{aligned}
$$

# Transfer function

Using the shift operator, $qx(kT) = x(kT + T)$, we can write

$$\begin{aligned} x(kT + T) &= Fx(kT) + Gu(kT) \\ y(kT) &= Hx(kT) \end{aligned}$$

as

$$\begin{aligned} qx(kT) &= Fx(kT) + Gu(kT) \\ y(kT) &= Hx(kT) \end{aligned}$$

This gives

$$y(kT) = H(qI - F)^{-1}Gu(kT)$$

**The transfer function** $G_T(z) = H(zI - F)^{-1}G$ (med $z \in \mathbb{C}$) gives an input-output description of the model.

# Transfer Functions...

A general rational transfer function (without direct term)

$$G_T(q) = \frac{B(q)}{A(q)} = \frac{b_1 q^{n-1} + \ldots + b_n}{q^n + a_1 q^{n-1} + \ldots + a_n}$$

corresponds to a **difference equation** $y(kT) = \frac{B(q)}{A(q)} u(kT)$, i.e.

$$y((k+n)T) + \ldots + a_n y(kT) = b_1 u((k+n-1)T) + \ldots + b_n u(kT)$$

Alternative notation: $G_T(q) = \sum_{m=1}^{\infty} g_T(m) q^{-m}$, where $g_T(m)$ is the **impulse response**.

# Poles

- Continuous-time minimal (controllable and observable) state-space model:

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \end{cases}$$

  **Poles** $=$ eigenvalues $\lambda_i$ to the matrix $A$
- The corresponding sampled model has (if it is minimal) poles that are equal to the eigenvalues $e^{\lambda_i T}$ to the matrix $F = e^{AT}$.
- The **stability region** for a discrete-time model is the interior of the unit circle

# Poles: Observations

Assume that $\lambda = \mu + i\omega \Rightarrow e^{\lambda T} = e^{\mu T}(\cos(\omega T) + i \sin(\omega T))$

$T$ small $\Rightarrow$ poles close to $z = 1$

$\mu < 0 \Rightarrow |e^{\lambda T}| < 1$ (stability is preserved)

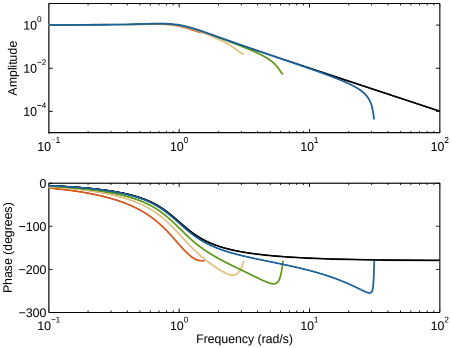
# Zeros

- No simple relation between the zeros of the continuous-time and discrete-time models
- The sampled model can have more zeros than the continuous-time model

# Frequency Response

Frequency response in continuous time: $G(i\omega)$
Frequency response in discrete time: $G_T(e^{i\omega T})$

Frequency responses for a continuous-time model and the corresponding sampled models for $T = 2$s, $T = 1$s, $T = 0.5$s, $T = 0.1$s:



# Signals and Disturbances

# Modeling of Signals

- Generic **signal models**: The signal is described as the output of a model (e.g, state-space or transfer function models) that has a standard signal (e.g., impulse, multisine, white noise) as input
- First-principles modeling (physical modeling) can be used to obtain signal models provided that he underlying mechanisms are known
- Data-driven modeling can be used to obtain signal models if data are available
- Stochastic descriptions: expected value, covariance function, spectral density
- Spectral density: Signal energy or power as a function of frequency
- A disturbance is an input signal to a system that we cannot choose or influence ourselves. This class of signals is particularly interesting to model.

# Example: A Common Class of Signal Models

One common example: Model $w(t)$ as the output of a linear model with a white noise process (all signal components are independent) as input:

$$D(q)w(kT) = C(q)e(kT) \quad \Leftrightarrow \quad w(kT) = G(q)e(kT), \quad \text{where } G(q) = \frac{C(q)}{D(q)}$$

($C(q)$ and $D(q)$ are polynomials)

- $G(q) = C(q)$: Moving average (MA) model (or process if referring to $w$)
- $G(q) = \frac{1}{D(q)}$: Auto-regressive (AR) model (process)
- $G(q) = \frac{C(q)}{D(q)}$: Auto-regressive moving average (ARMA) model (process)

# Stochastic Modeling

A **stochastic process** is a sequence of random variables

$$w(t_1),\ w(t_2),\ w(t_3),\ \dots$$

- Assume uniform sampling $t_k = kT$
- The mean value function: $m_w(t) = \mathrm{E}(w(tT))$
- The covariance function:

$$R_w(t,s) = \mathrm{Cov}(w(t), w(s)) = \mathrm{E}((w(tT) - m_w(t))(w(sT) - m_w(s)))$$

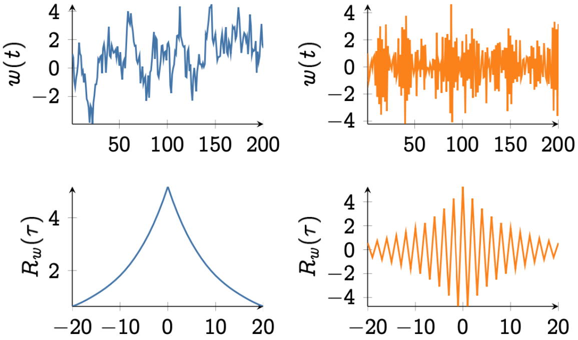
For a stationary process:

- The mean value does not depend on $t$
- $R_w(t,s)$ depends only on $t-s$ and can be replaced with

$$R_w(\tau) = \mathrm{E}((w(tT) - m_w)(w((t-\tau)T) - m_w))$$

# Example

Realizations of two stochastic processes and their covariance functions:



# Spectral Density

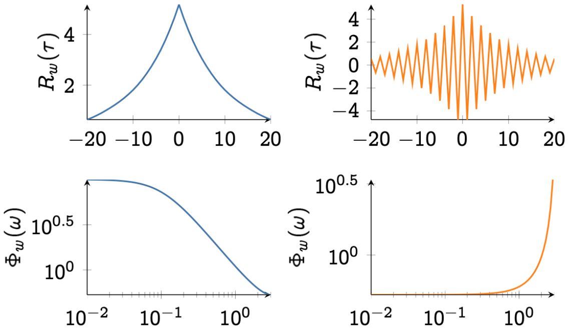
The **spectral density** (sometimes called spectrum) $\Phi_w(\omega)$ of a signal can be defined as:

$$\Phi_w(\omega) = T \sum_{k=-\infty}^{\infty} R_w(\tau) e^{-i\omega kT}$$

Similar definitions are available also for deterministic signals (continuous time or discrete time, finite energy or infinite energy), see Appendix D.

# Example

Two covariance functions and the corresponding spectral densities:



# Cross-Covariance and Cross-Spectral Density

Similarities and dependencies between signals can be described using the cross-covariance function

$$R_{yu}(\tau) = \mathrm{E}((y(t) - m_y)(u(t-\tau) - m_u))$$

and the cross-spectral density

$$\Phi_{yu}(\omega) = T \sum_{k=-\infty}^{\infty} R_{yu}(\tau) e^{-i\omega kT}$$

Uncorrelated signals $\Leftrightarrow R_{yu} = \Phi_{yu} = 0$

# Spectral Densities and Linear Filtering

Consider a linear model

$$y = Gu + w$$

where $u$ and $w$ are uncorrelated. Then

$$\begin{aligned} \Phi_y(\omega) &= |G(e^{i\omega T})|^2 \Phi_u(\omega) + \Phi_w(\omega) \\ \Phi_{yu}(\omega) &= G(e^{i\omega T}) \Phi_u(\omega) \end{aligned}$$

(Same expressions in continuous time if $e^{i\omega T}$ is replaced with $i\omega$)

# Summary

**Sampling and Discrete-Time Models**

- Sampling
- Aliasing and the sampling theorem
- Sampling of a linear state-space model
- Discrete-time linear models (z transform, transfer function, impulse response, difference equation, poles, zeros, frequency response)

**Signals and Disturbances:**

- Signal (and in particular disturbance) models
- MA, AR, ARMA models
- Stochastic processes
- Covariance functions
- Spectral densities

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