Modeling and Learning for Dynamical Systems

Lecture 2

Martin Enqvist

LINKÖPING UNIVERSITY

Principles for Model Building

The Three Phases of Modeling

- **Phase 1: The problem is structured:** Which variables are important? How do they affect each other? Split into subsystems. A block diagram might give a useful overview.
- **Phase 2: The basic equations are formulated:** Basic equations for the subsystems based on physical laws. Dimensions. Approximations. Idealizations.
- **Phase 3: Rewrite and structure the equations:** Organize the equations and their relationships. Develop a state-space model (or alternatively a DAE model).

Dimensions

Physical variables have **dimensions** (here denoted by square brackets). For example:

$$[\text{density}] = \frac{[\text{mass}]}{[\text{length}]^3} = ML^{-3}$$

$$[\text{force}] = \frac{[\text{mass}] \cdot [\text{length}]}{[\text{time}]^2} = MLT^{-2}$$

quantity	dimension	unit
length	L	meter, m
time	T	second, s
mass	M	kilogram, kg
velocity	LT^{-1}	m/s
acceleration	LT^{-2}	m/s^2
area	L^2	m^2
volume	L^3	m^3
density	ML^{-3}	kg/m^3
force	MLT^{-2}	Newton, N
pressure	$ML^{-1}T^{-2}$	Pascal, Pa
energy	ML^2T^{-2}	Joule, J
power	ML^2T^{-3}	watt, W
viscosity	$ML^{-1}T^{-1}$	$Pa\ s$

Example: Bernoulli's Law

Bernoulli's law (relating outflow speed v and tank level h): $v = \sqrt{2gh}$
Left hand side dimension: [velocity] $= LT^{-1}$
Right hand side dimension: $[\sqrt{2gh}] = \sqrt{LT^{-2} \cdot L} = LT^{-1}$
(matches left hand side, OK)

Physical variables are measured in units (which are not always the same). A change of unit ⇒ a scaling of the affected variables:

- changed length unit: length variable has to be scaled with a factor $\lambda > 0$
- changed time unit: time variable has to be scaled with a factor $\nu > 0$

The new variables $\lambda\nu^{-1}v$, $\lambda\nu^{-2}g$, λh still satisfy Bernoulli's law:

$$\lambda\nu^{-1}v = \sqrt{2(\lambda\nu^{-2}g)(\lambda h)} \quad \Leftrightarrow \quad v = \sqrt{2gh}$$

Bernoulli's Law...

Bernoulli's law can be rewritten as

$$\frac{v^2}{gh} = 2$$

The variable $\pi = \frac{v^2}{gh}$ is **dimensionless**:

$$[\pi] = \frac{(LT^{-1})^2}{(LT^{-2})(L)} = L^0T^0$$

Scaling

- Laws of nature should hold no matter the choice of units $\Rightarrow$ Restriction of which functions that can occur (they are called **dimensionally homogenous**)
- **Dimensionsless variables** are particularly convenient since they are unaffected by unit scalings and equations involving only such variables are therefore always dimensionally homogenous
- The Mach number for an aircraft is one example of a dimensionless variable
- **Buckingham's theorem:** All dimensionally homogenous equations can be rewritten as equations involving only dimensionless variables
- Scaling can be used to reduce the number of experiments or simulations

Example: Scaling

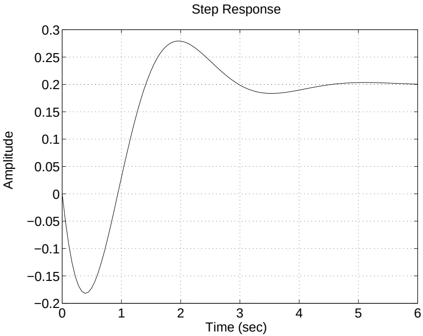
The figure shows the step response for the model

$$\ddot{y} + 2\dot{y} + 5y = -\dot{u} + u$$

Use this to sketch the step response for the model

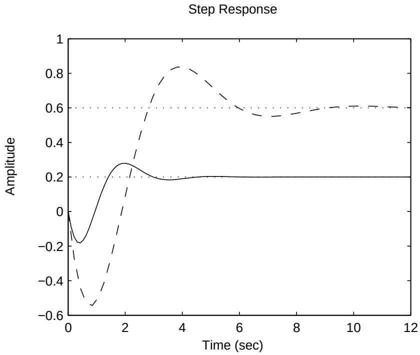
$$4\ddot{y} + 4\dot{y} + 5y = -6\dot{u} + 3u$$

without performing a new simulation.



Example: Scaling

The figure shows the original step response (solid) and the rescaled one for $\alpha = 3$ and $\beta = 2$

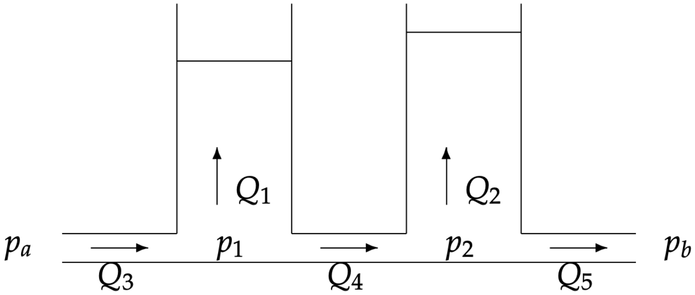


Simplified Models

Models always contain simplifcations. In particular:

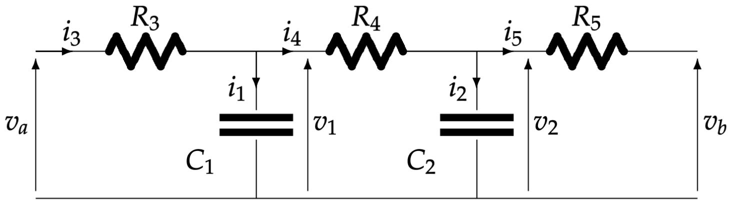
- **Small effects are neglected**
  - Always done to some extent
  - Examples: mass-less subsystems, no friction, incompressible liquid, linearity, perfectly mixed liquid or gas, ideal gas
- **Separation of time constants**
  - Focus on modeling phenomena whose time constants match the intended use of the model
  - Approximate considerably faster subsystems with static relationships
  - Approximate considerably slower subsystems with constants
- **Aggregation of state variables**
  - Merge several similar state variables into one

Example: A Flow System



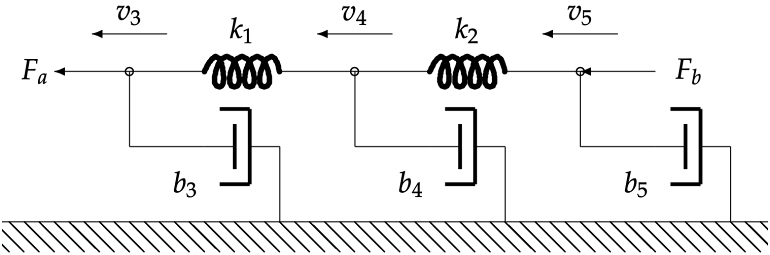
- Two tanks in series
- The fluid is incompressible
- Pressures: p_x
- Volume flows: Q_x

Example: An Electric Circuit



- Voltages: v_x
- Currents: i_x
- Resistors: R_x
- Capacitors: C_x

Example: A Mechanical System



- Forces: F_x
- Velocities: v_x
- Friction constants in dampers: b_x
- Spring constants: k_x

Similarities between Domains

- There are many similarities between different physical domains (electric circuits, mechanics, hydraulics, heat systems).
- Most fundamental equations define relationships between **effort variables** (voltage, force, torque, pressure, temperature) and **intensity variables** (current, velocity, angular velocity, volume flow, heat flow).
- Furthermore, components like capacitors, springs, tanks and heat storage units have similar functions in the domains.
- Similarly, resistance and friction have corresponding roles as well as inductors and inertia.
- Table 5.3 presents an overview of these physical analogies.

Summary

- The three phases of modeling
- Dimensions and scaling
- Dimensionless variables and Buckingham's theorem
- Simplified models
- Similarities between domains

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