Solutions to exam in TSRT92 Modelling and Learning for Dynamical Systems

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Martin Enqvist

1. (a) Choosing

$$H(q) = \frac{1}{1 + a_1 q^{-1} + a_2 q^{-2}}$$

results in an ARX model.

(b) Since $R_e(\tau) = \lambda_e \delta(\tau)$ and

$$\begin{aligned} R_y(\tau) &= E\left[y(t)y(t-\tau)\right] \\ &= E\left[(b_1 e(t-1) + \ldots + b_N e(t-N))(b_1 e(t-\tau-1) + \ldots + b_N e(t-\tau-N))\right] \end{aligned}$$

we have $R_y(\tau) = 0$ if $|\tau| > N-1$, i.e., $M = N-1$.

(c) Since $\Phi_u(\omega) = 1$, $\Phi_e(\omega) = 2$ and $\Phi_{ue}(\omega) = 0$ we have

$$\Phi_y(\omega) = |G(e^{i\omega})|^2 \Phi_u(\omega) + \Phi_e(\omega) = (1 + 0.5e^{-i\omega})(1 + 0.5e^{i\omega}) + 2 = 3.25 + \cos(\omega)$$

(d) An OE model corresponds to a noise model $H(q) = 1$ but the prior information in this case indicates that the true system has a more complex noise model $H_0(q) \neq 1$. Because of this mismatch, an OE model estimated directly with the prediction-error method will usually be (asymptotially) biased when data have been collected in closed loop. Hence, this combination of model structure and method is not a good idea under these circumstances. Instead, it would be useful to either consider other model structures such as ARX, ARMAX or BJ or to use the two-stage method instead of the direct prediction-error approach to estimate an OE model.

2. (a) Let $\tau = \beta t$, $z = \alpha y$ and $v = u$. Then

$$y(t) = \frac{1}{\alpha} z(\tau) = \frac{1}{\alpha} z(\beta t)$$

Differentiating gives

$$\begin{aligned} \frac{dy(t)}{dt} &= \frac{1}{\alpha}\frac{dz(\beta t)}{dt} = \frac{\beta}{\alpha}\frac{dz(\tau)}{d\tau} \\ \frac{d^2y(t)}{dt^2} &= \frac{d}{dt}\left(\frac{dy(t)}{dt}\right) = \frac{d}{dt}\left(\frac{\beta}{\alpha}\frac{dz(\beta t)}{d\tau}\right) = \frac{\beta^2}{\alpha}\frac{d^2z(\tau)}{d\tau^2} \\ \frac{du(t)}{dt} &= \beta\frac{dv(\tau)}{d\tau} \end{aligned}$$

Inserting this in the original model and multiplying both sides with 3 gives

$$\frac{3\beta^2}{\alpha} z'' + \frac{6\beta}{\alpha} z' + \frac{15}{\alpha} z = -3\beta v' + 3v$$

We get the second model equation by selecting $\alpha = 3$ and $\beta = 2$. Hence, the previous simulation can be reused to give the step response also of the second model by scaling the output with $\alpha = 3$ and rescaling the time axis with $\beta = 2$.

(b) Stiff models are difficult because they contain both fast and slow modes, which makes it harder to vary the step length without stability or accuracy problems. One simple way to quantify the stiffness of a (linear) state-space model is to look at its eigenvalues. The eigenvalues of the 3 models are $\{-1, -2\}$, $\{-0.001, -1\}$, and $\{-1, -100\}$. The ratio between them is 2, 1000 and 100. Hence (i) is the least stiff model, followed by (iii), while (ii) is the stiffest.

(c) At stationarity, with a constant input, all signals are constant and their derivatives are zero. This gives $0 = -2x_3$ and $0 = x_1 + 3x_3 + 2u$, i.e., $x_3 = 0$ and $x_1 = -2u$. Inserting this and $y = x_2$ in the algebraic equation gives

$$0 = -2u + y - u \quad \Rightarrow \quad y = 3u$$

i.e., the static gain is 3.

3. (a) Data is first imported into the GUI and split in two equal parts which are used as estimation and validation data. The ARX, ARMAX, OE and BJ classes of models give all good fit ($> 75\%$). The one selected here a BJ(3,1,1,3,1) model which has the highest fit (83.9% for validation data), although its residuals plot is not perfect. As comparison, an OE(4,4,1) model is also included for comparison in some of the plots.

```
bj31131 =
Discrete-time BJ model: y(t) = [B(z)/F(z)]u(t) + [C(z)/D(z)]e(t)

B(z) = 0.259 (+/- 0.01631) z^-1 + 0.4954 (+/- 0.02223) z^-2
       - 0.03493 (+/- 0.02454) z^-3
C(z) = 1 + 0.8684 (+/- 0.04612) z^-1
D(z) = 1 + 0.9532 (+/- 0.02832) z^-1
F(z) = 1 - 0.6586 (+/- 0.01031) z^-1 + 0.9263 (+/- 0.002202)
       z^-2 - 0.6888 (0.009323) z^-3

Name: bj31131
Sample time: 1 seconds
Fit to estimation data: 86.82% (prediction focus)
FPE: 0.02728, MSE: 0.02642
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Some comments:

* The parameter uncertainty is small, which can be seen by comparing the standard deviations (in parentheses) with the parameter estimates.
* The simulated model output for validation data is shown for both models in Figure 1. The BJ and OE models have similar model fit values, with a slight advantage for the BJ model. The largest errors seem to be at the peaks, where there seems to be a saturation in the true system which the linear models naturally cannot describe.
* The residuals from both models are shown in Figure 2 and are not perfectly white, but the BJ model is clearly better than the OE model. Increasing the orders for the noise model can improve the whiteness of the residuals, but due to the limitation given in the exercise, we would then have to reduce the orders of the input-output part which results in a worse model fit value. Hence, this choice is not recommended here.
* The Bode plot and the zeros and poles of the BJ(3,1,1,3,1) model are shown in Figure 3. The uncertainty of the frequency response is small (small confidence intervals) which indicates that the model can be trusted in the whole frequency range. Furthermore, the confidence intervals for the poles and zeros are also relatively small and the poles are clearly separated from the zeros, which indicates that the model order is not too high. All poles are stable (within the unit circle).

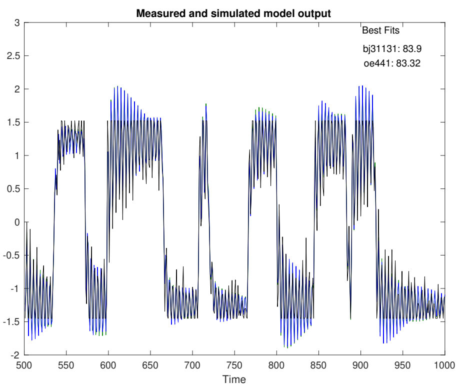

Figure 1: Simulated model output for the BJ(3,1,1,3,1) (green) and OE(4,4,1) (blue) models and measured output (black).

(b) As can be seen by looking at y, the output of the system shows a saturation. This can be modeled using a Wiener model (which is a special case of a Hammerstein-Wiener model). Here, a Wiener model with four poles and zeros in the linear part and a piecewise linear output nonlinearity with 3 units has been estimated and the simulated output for validation data is shown in Figure 4. The model fit has increased to 88.2% from 83.9% for the BJ(3,1,1,3,1) model. The estimated frequency response of the linear part and the output nonlinearity are

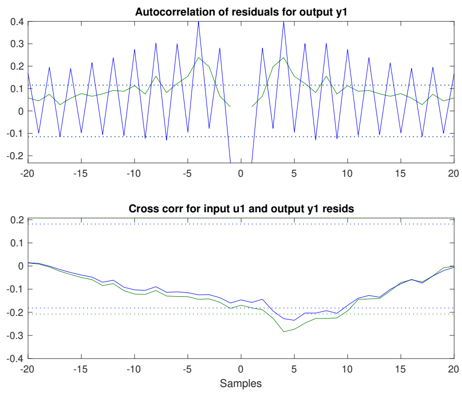

Figure 2: Residuals for the BJ(3,1,1,3,1) (green) and OE(4,4,1) (blue) models.

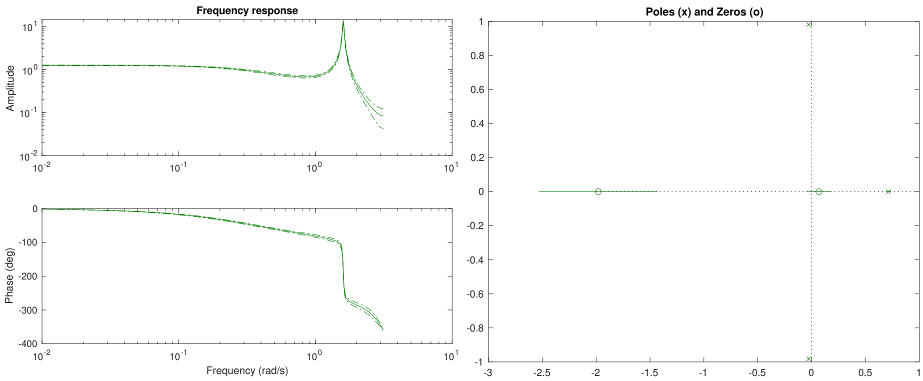

Figure 3: Bode plot (left) and zero and pole locations (right) for the BJ(3,1,1,3,1) model.

shown in Figure 5. The frequency response is very similar to the one for the BJ(3,1,1,3,1) model and the nonlinearity is clearly a saturation.

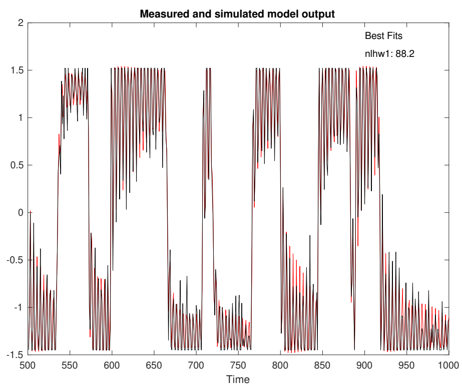

Figure 4: Simulated model output for the Wiener model (red) and measured output (black).

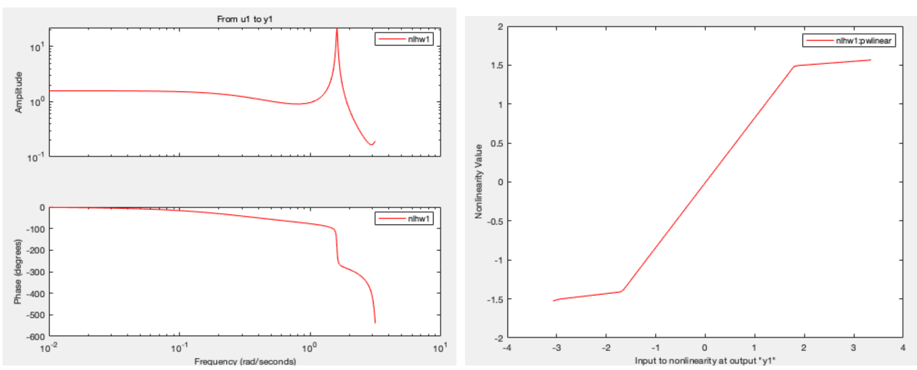

Figure 5: Bode plot and output nonlinearity for the estimated Wiener model.

4. (a) Let $\theta = \begin{pmatrix} b_1 & b_2 \end{pmatrix}^T$. The predicted output for the given FIR model is

$$\hat{y}(t|\theta) = b_1 u(t-1) + b_2 u(t-2)$$

and he prediction error can be written

$$\varepsilon(t,\theta) = y(t) - \hat{y}(t) = (0.2 - b_1)u(t-1) + (0.4 - b_2)u(t-2) + 0.5u(t-3) + w(t),$$

The cost function $V_N(\theta)$ will converge to

$$\begin{aligned}\overline{V}(\theta) = &\operatorname{E}\varepsilon(t,\theta)^2 = \left((0.2-b_1)^2 + (0.4-b_2)^2 + 0.25\right) R_u(0) + \lambda_w \\ &+ 2\left((0.2-b_1)(0.4-b_2) + 0.5(0.4-b_2)\right) R_u(1) + (0.2-b_1)R_u(2) \\ &+ 2(0.2-b_1)R_{wu}(1) + 2(0.4-b_2)R_{wu}(2) + R_{wu}(3).\end{aligned} \tag{1}$$

when $N \to \infty$. Here, we have $R_u(0) = \lambda_u$ and $R_w(0) = \lambda_w$. Since $R_{wu}(k) = 0$ for all k and $R_u(\tau) = \delta(\tau)\lambda_u$, we can simplify (1) to

$$\overline{V}(\theta) = \left((0.2-b_1)^2 + (0.4-b_2)^2 + 0.25\right)\lambda_u + \lambda_w$$

The minimum of $\overline{V}(\theta)$ can be found from

$$\frac{\partial V(b)}{\partial b} = 2\lambda_u \begin{pmatrix} 0.2 - b_1 \\ 0.4 - b_2 \end{pmatrix} = 0$$

and the result is $\hat{b}_1 = b_{1,0} = 0.2$ and $\hat{b}_2 = b_{2,0} = 0.4$. This means that the estimated parameters b_1 and b_2 converge to their true values even though the model is not unbiased since it cannot describe the true system for any value of b_i due to the missing term $b_3 u(t-3)$ in the model.

(b) A nonwhite w does not affect the parameter convergence since $R_w(\tau)$ for $\tau \neq 0$ is not present in (1).

(c) A nonwhite u will change the asymptotic cost function to

$$\begin{aligned}\overline{V}(\theta) = &\operatorname{E}\varepsilon(t,\theta)^2 = \left((0.2-b_1)^2 + (0.4-b_2)^2 + 0.25\right) R_u(0) + \lambda_w \\ &+ 2\left((0.2-b_1)(0.4-b_2) + 0.5(0.4-b_2)\right) R_u(1) + (0.2-b_1)R_u(2)\end{aligned}$$

and the minimum is now defined by

$$\frac{\partial V(b)}{\partial b} = \begin{pmatrix} 2\lambda_u(0.2-b_1) - 2(0.4-b_2)R_u(1) - R_u(2) \\ 2\lambda_u(0.4-b_2) - 2(0.2-b_1-0.5)R_u(1) \end{pmatrix} = 0$$

Since this equation is not solved by $b_1 = 0.2$ and $b_2 = 0.4$, we can conclude that the estimated parameters do not converge to the true values in this case.

(d) With a nonwhite w and closed-loop data, all terms in (1) are in general nonzero. Hence, similarly to the previous exercise, we can conclude that the estimated parameters will not converge to their true values. This is in line with the general result about direct closed-loop identification stating that the true system must be possible to describe with the model structure for the parameter estimates to be asymptotically unbiased in closed loop.

5. (a) Differentiating equation (3) from the exam gives

$$\dot{x}_1 x_3 + x_1 \dot{x}_3 - \dot{x}_2(2 + x_3) - x_2\dot{x}_3 + 1 = 0$$

and inserting equations (1) and (2) from the exam results in

$$(1 + x_2)x_3 - \dot{x}_2(2 + x_3) + 1 = 0 \tag{2}$$

Since $\dot{x}_3 = 0$, we have that $x_3(t) = x_3(0) =$ constant for all t. There are two cases to consider.

Case 1: If $x_3 \neq -2$, then one can solve (2) for $\dot{x}_2$ which gives

$$\dot{x}_2 = \frac{1 + x_3 + x_2 x_3}{2 + x_3}$$

and hence the index is 1.

Case 2: If instead $x_3 = -2$ then (2) becomes

$$x_3 + x_2 x_3 + 1 = 0$$

which can be differentiated again, leading to

$$-2\dot{x}_2 = 0$$

and therefore the index is 2 in this case.

(b) Differentiating the algebraic constraint (4) from the exam gives

$$\dot{x}_1 x_3 + x_1 \dot{x}_3 - 2\dot{x}_2 + 1 = 0$$

or

$$\dot{x}_2 = \frac{1}{2} + \frac{(1 + x_2)x_3}{2}$$

In this case, the index is always 1, and there are no singular points. Hence, the index differs from Case 2 above.