Solution for the exam in the course “Modeling and Learning for Dynamical Systems" (TSRT92) 2023-08-21

1. (a) Examples of information:
   - what variables are affected by the input
   - character of the response (oscillatory, unstable, damped, etc.)
   - stationary gain
   - time constant
   - time delay
   - settling time

   (b) Let $x_1 = y$ and $x_2 = \dot{y}$, this yields

$$\begin{aligned}\dot{x}_1 &= \dot{y} = x_2 \\ \dot{x}_2 &= \ddot{y} = -\frac{\dot{y}y^3}{1+\dot{y}y^2} - \frac{y}{1+\dot{y}y^2}u = -\frac{x_1^3 x_2}{1+x_1^2 x_2} - \frac{x_1}{1+x_1^2 x_2}u\end{aligned}$$

   (c) If we consider the test equation

$$\dot{x} = f(x) = \lambda x$$

   then we have:

$$x_n = x_{n-1} + hf(x_n) = x_{n-1} + h\lambda x_n$$

   i.e.,

$$x_n = \frac{1}{1-h\lambda}x_{n-1} = \frac{1}{(1-h\lambda)^n}x_0$$

   which converges to zero (i.e., $\lim_{t\to\infty} x_n = 0$) if $|1-h\lambda| > 1$, that is $h\lambda$ lies outside a disk of radius 1 centered at 1. In particular, then, the numerical method is stable on the entire left half plane.

   If we look at the system $\dot{x} = 10x$, then $\lambda = 10$, but in this case the system is *unstable*, and to reproduce its qualitative behavior we have to impose that the numerical method is also unstable, i.e., that $|1-10h| < 1$ or $h < 0.2$.

   (d) The system is in a standard linear state space form

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

   Its time constants corresponds to the modes of the system, and can be computed from the characteristic equation:

$$\det(\lambda I - A) = \begin{bmatrix}\lambda+0.8 & 0 & 0 \\ 0 & \lambda+1.3 & 0 \\ 0 & 0 & \lambda+20\end{bmatrix} = 0$$

i.e., $\lambda = \{-0.8, -1.3, -20\}$. Since A diagonal, the modes are decoupled, and the fastest dynamics is immediately recognized as the one of x_3. Putting $\dot{x}_3 = 0$ gives

$$x_3(t) = 0.7u(t)$$

The requested approximation is therefore

$$\begin{aligned}\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}-0.8 & 0 \\ 0 & -1.3\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}1 \\ 2\end{bmatrix}u \\ y &= \begin{bmatrix}2 & 4\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + 0.35u\end{aligned}$$

2. For the model

$$y(t) = b_1 u(t-1) + b_2 u(t-2) + e(t)$$

the predictor is

$$\hat{y}(t|\theta) = b_1 u(t-1) + b_2 u(t-2)$$

with

$$\theta = \begin{bmatrix}b_1 \\ b_2\end{bmatrix}$$

To compute the covariance of the estimates we use (12.93) of the book (Ljung, Glad, & Hansson, 2021)

$$P_N = E(\hat{\theta}_N - \theta_0)(\hat{\theta}_N - \theta_0)^T \approx \frac{1}{N}\lambda_e \bar{R}^{-1}$$

where

$$\bar{R} = E\psi(t,\theta)\psi^T(t,\theta)$$

and ψ is the gradient

$$\psi(t,\theta) = \frac{d}{d\theta}\hat{y}(t|\theta) = \begin{bmatrix}u(t-1) \\ u(t-2)\end{bmatrix}$$

(a) If u is a white noise of variance r, then

$$\bar{R} = E\begin{bmatrix}u(t-1) \\ u(t-2)\end{bmatrix}\begin{bmatrix}u(t-1) & u(t-2)\end{bmatrix} = \begin{bmatrix}r & 0 \\ 0 & r\end{bmatrix}$$

hence

$$P_N \approx \frac{1}{Nr}I_2$$

(b) If u has covariance $R_u(\tau) = a^{|\tau|}$ then

$$\bar{R} = \begin{bmatrix}1 & a \\ a & 1\end{bmatrix}$$

hence

$$P_N \approx \frac{1}{N(1-a^2)}\begin{bmatrix}1 & -a \\ -a & 1\end{bmatrix}$$

(c) For case b):

$$E(\hat{b}_1 - b_{10})^2 = \frac{1}{N(1-a^2)}$$

and

$$\begin{aligned} E(\hat{b}_1 + \hat{b}_2 - b_{10} - b_{20})^2 &= E(\hat{b}_1 - b_{10})^2 + 2E(\hat{b}_1 - b_{10})(\hat{b}_2 - b_{20}) + E(\hat{b}_2 - b_{20})^2 \\ &= \frac{1}{N(1-a^2)} - \frac{2a}{N(1-a^2)} + \frac{1}{N(1-a^2)} \\ &= \frac{2}{N(1+a)} \end{aligned}$$

3. System identification exercise

(a) First of all notice that non-parametric identification suggests that the system transfer function could have a resonance around 2 rad/sec, see Fig. 1 This is confirmed

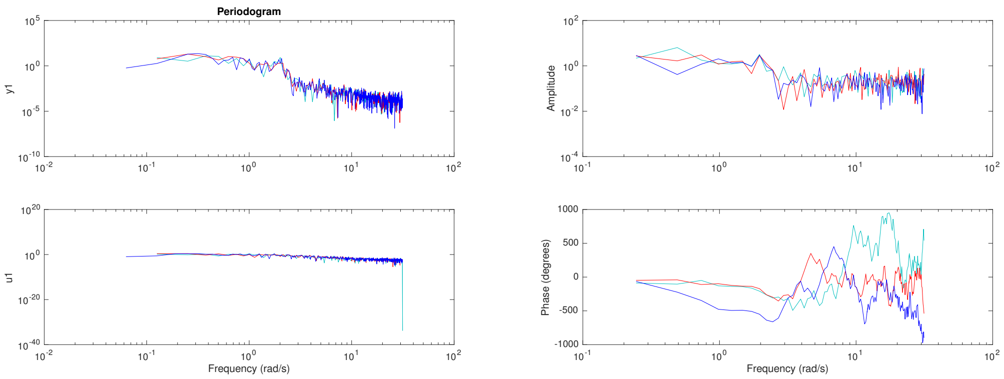


Figure 1: Periodogram to the left, ETFE to the right.

on the data themselves: look for example at the interval between 50 and 60 sec ("step response" shows oscillatory pattern), see Fig. 2.

(b) The ARX order selection tool of the System Id toolbox suggests ARX(10,8,6) (fit of 41.7% to estimation data, blue plots). Much better can be done with less parameters. For instance the two models OE(4,4,1) (fit of 51.3%, green) and BJ(3,3,3,3,1) (fit of 50.6%, red) are shown below.

```
oe441 =
Discrete-time OE model:  y(t) = [B(z)/F(z)]u(t) + e(t)
  B(z) = -0.05118 z^-1 + 0.1607 z^-2 - 0.1692 z^-3 + 0.06074 z^-4

  F(z) = 1 - 3.704 z^-1 + 5.177 z^-2 - 3.232 z^-3 + 0.759 z^-4

Sample time: 0.1 seconds

Parameterization:
```

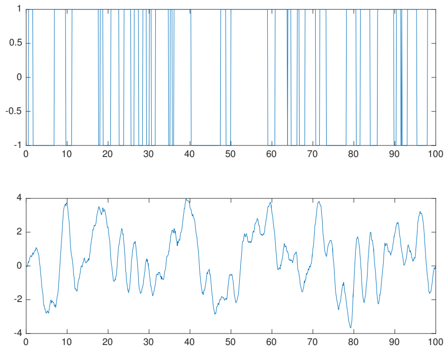
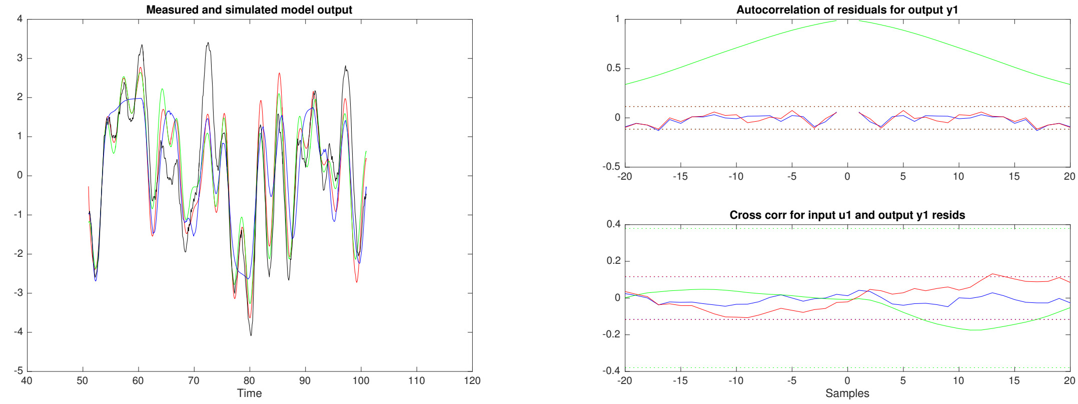


Figure 2: The input and output of the data.

```
   Polynomial orders:   nb=4   nf=4   nk=1
   Number of free coefficients: 8

Status:
Estimated using PEM on time domain data "mydatade".
Fit to estimation data: 50.37% (simulation focus)
FPE: 0.832, MSE: 0.8058


bj33331 =
Discrete-time BJ model:  y(t) = [B(z)/F(z)]u(t) + [C(z)/D(z)]e(t)
  B(z) = 0.01912 z^-1 - 0.04211 z^-2 + 0.02622 z^-3

  C(z) = 1 + 0.2141 z^-1 - 0.7591 z^-2 + 0.02683 z^-3

  D(z) = 1 - 0.933 z^-1 - 0.9841 z^-2 + 0.9284 z^-3

  F(z) = 1 - 2.913 z^-1 + 2.867 z^-2 - 0.9533 z^-3

Sample time: 0.1 seconds

Parameterization:
   Polynomial orders:   nb=3   nc=3   nd=3   nf=3   nk=1
   Number of free coefficients: 12

Status:
Estimated using PEM on time domain data "mydatade".
Fit to estimation data: 94.16% (prediction focus)
FPE: 0.01171, MSE: 0.01116
```

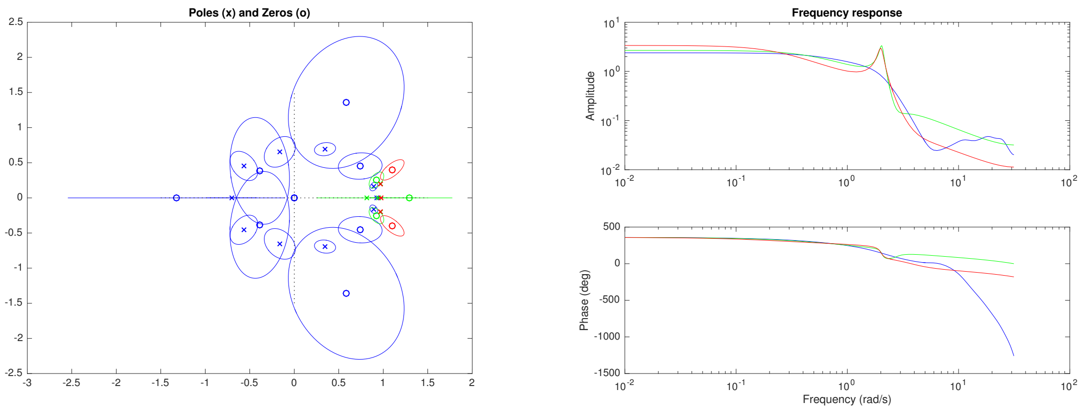
The time-domain fit of the 3 models is shown in Fig. 3(a). The ARX model does not follow the oscillatory pattern (again look for instance at the interval 50 sec - 60 sec), hint that it does not reproduce the resonance correctly. The other two seem better. This is confirmed by the frequency response plots, see Fig. 4(b). Residuals



(a) Time-domain fit (b) Residuals

Figure 3: Black: true data. Blue: ARX. Green: OE. Red: BJ

are in Fig. 3(b), and show that for OE the residuals are not white, hence the model is not adequate. Zeros/poles are in Fig. 4(a). Confidence intervals for ARX are inadequate (too much overlap), hence the model is not a good one. The frequency



(a) Pole-zero plot (b) Frequency responses

Figure 4: Blue: ARX. Green: OE. Red: BJ

responses are shown in Fig. 4(b) and confirm the presence of a resonance peak around 2 rad/sec.

In summary of the 3 models considered here the only good one is BJ(3,3,3,3,1). Notice how failure of OE to give white residuals is a hint that the transfer function for the disturbance is a nontrivial one.

4. (a) When E is non-singular it is invertible, hence one can write

$$\dot{x} = -E^{-1}Fx + E^{-1}Gu = Ax + Bu$$

E is non-singular when $\det E \neq 0$ i.e., when $\alpha \neq -1$. Hence $\alpha_o = -1$.

(b) When $\alpha \neq -1$ the index is 0 (the system is an ODE not a DAE). When $\alpha = \alpha_o = -1$, then subtracting the first row and adding the second row to the third row one gets

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} x = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} u \tag{1}$$

Differentiating the third row and summing it to the rest

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \dot{x} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix} \dot{u}$$

the matrix in front of $\dot{x}$ has full rank, hence the index in this case is 1.

(c) Considering again the transformation (1), from the third row we have $x_1 - x_2 + x_3 = 2u_2$, from which x_3 can be obtained, and replaced in the two other equations, which are as follows

$$\begin{aligned} \dot{x}_1 + \dot{x}_3 + x_2 &= u_1 \\ \dot{x}_1 - \dot{x}_2 + x_3 &= u_1 + u_2 \end{aligned}$$

leading to

$$\begin{aligned} \dot{x}_2 &= -x_2 + u_1 - 2\dot{u}_2 \\ \dot{x}_1 - \dot{x}_2 &= x_1 - x_2 + u_1 + u_2 \end{aligned}$$

replacing $\dot{x}_2$ in the second equation with its expression from the first equation we get the desired form, with

$$A_1 = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & -2 \\ 0 & -2 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 2 \end{bmatrix}$$