Solution for the exam in the course “Modeling and Learning for Dynamical Systems" (TSRT92) 2023-01-05

1. (a) A model is never a true description of a system, but is developed to be useful in solving a problem. This “usefulness” is called the validity of the model with respect to the purpose in question. Model validation is thus establishing that a particular model is valid. (2p)

   (b) A simulation of a model is based on the mathematical description and uses numerical methods to generate an approximate solution. The stability range of a differential equation does not necessarily correspond to the stability range of the numerical algorithm, which must be kept in mind during the simulation. (2p)

   (c) The stationary points are given by $\dot{x} = f(x_0, u_0) = 0$. For this to be true, $x_{2,0} = x_{3,0} = 0$. Furthermore, it must be true that $\sin x_1 = 0$, which is the case when $x_{1,0} = \pi n$, where n is an integer. (2p)

   (d) When solving the system for the internal variables, the higher the index of the DAE, the more times the input signal is typically differentiated. Since differentiation is sensitive to numerical errors, it then becomes difficult to get good accuracy in the solution. (2p)

   (e) The model he simulates probably has both fast and slow time constants, resulting in a stiff system. Linus should concentrate on accurately modelling the dynamics of the domain that he is mainly interested in and replace significantly slower and faster dynamics with approximate relationships. (2p)

2. (a) There are a lot of questions and considerations that need to be taken into account before designing your model. Here are a few of them, divided into three “phases”.
   Experiment design: What are the goals of my model, what questions should it answer? How much do I know in advance about my system? Do initial experiments need to be done to find out the appropriate operating point, linearity/non-linearity, sampling time, etc.? Which signals to measure, choice of input signal to excite the system sufficiently, possible post-processing of data, is it an identification experiment under feedback?
   Model structure: Try with a simple model first, typically a low order linear black-box model and see if it is good enough. Otherwise, you can think further about whether linear/non-linear model structure is the best fit, tailored or black-box model, parametric or non-parametric model and appropriate number of orders.
   Validation: Is cross-validation possible due to sufficient data, over-fitting, simulation/prediction, evidence of omission, confidence intervals for parameters? And finally the most important question: is the model good enough for its purpose? (3p)

   (b) The least squares estimate is given by $\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} = \left(\frac{1}{N} \sum \varphi(t)\varphi^T(t) \right)^{-1} \frac{1}{N} \sum \varphi(t) y(t)$,

   where the input variables are given by $\varphi(t) = \begin{pmatrix} u(t) \\ u(t-1) \end{pmatrix}$, $u(t) = \begin{cases} 2, & \text{t even} \\ 0, & \text{t odd} \end{cases}$

and finally $y(t) = 2 + \epsilon$ for each t.

This results in the estimate $\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} (2+\epsilon) = \begin{pmatrix} 1+\epsilon/2 \\ 1+\epsilon/2 \end{pmatrix}$.

(7p)

3. (a) The estimate $\hat{\theta}_N$ will converge to θ^* according to

$$\theta^* = \lim_{N\to\infty} \hat{\theta}_N = \arg\min_{\theta} \int_{-\pi}^{\pi} \mid G_0(e^{i\omega}) - G(e^{i\omega}, \theta) \mid^2 \Phi_u(\omega) d\omega$$

where $G_0(e^{i\omega})$ is the true system, $G(e^{i\omega}, \theta)$ is the model and $\Phi_u(\omega)$ is the signal spectrum. (The model of the controller is $H_*(e^{i\omega}) = 1$ here.) Thus, the model convergence is weighted with the input signal spectrum $\Phi_u(\omega)$.

Since we have the right model structure, there are values θ_0 such that $G_0(e^{i\omega}) = G(e^{i\omega}, \theta_0)$. Since the input signal is white noise, $\Phi_u(\omega)$ is constant and the result above gives that $\theta^* = \theta_0$, that is $\hat{a}_1 = -1.71$, $\hat{a}_2 = 0.79$, $\hat{b}_1 = 1$ and $\hat{b}_2 = 0.92$. (2p)

(b) Due to the constant input signal, the parameter convergence will be focused entirely on $\omega = 0$ because $\Phi_u(\omega) = 0$ for $\omega \neq 0$. That is,

$$\hat{b} = \arg\min_{b} \mid G_0(e^{i0}) - b \mid^2 = \frac{1 + 0.92}{1 - 1.71 + 0.79} \approx 24$$

(2p)

(c) We start by loading the dataset and studying its time and frequency characteristics (can also be done in the user interface by selecting `Time plot` or `Data spectra`):

```
load ex091012_4c
figure; plot(z1,z2,z3)
figure; plot(fft(z1),fft(z2),fft(z3))
```

We immediately see that the maximum amplitudes of the input signals are the same, but that the spectrum differs. `z1` contains all frequencies between 0–31 rad/s, `z2` approximately 0–2 rad/s and `z3` approximately 1.5–7 rad/s. The spectrum of `z1` is also significantly lower because the energy is distributed over more frequencies.

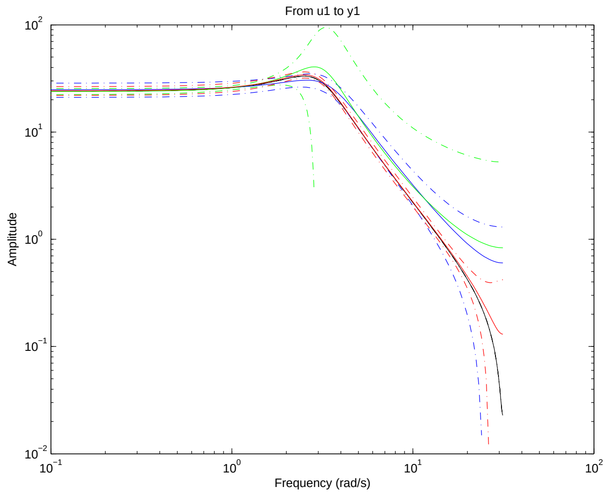
The models are estimated with:

```
m1=oe(z1,[2 2 1]);
m2=oe(z2,[2 2 1]);
m3=oe(z3,[2 2 1]);
```

If you look at the coefficients, it is above all b_1 and b_2 that varies a lot, but even the poles differ.

The models are evaluated above all in the frequency plane (`Frequency resp`) with confidence intervals plotted:

```
G0=idpoly(1,[0 1 0.92],1,1,[1 -1.71 0.79],[],0.1); % the true system
figure; bode(G0,'k',m1,'b',m2,'g',m3,'r','sd',3); % 3 standard deviations
```

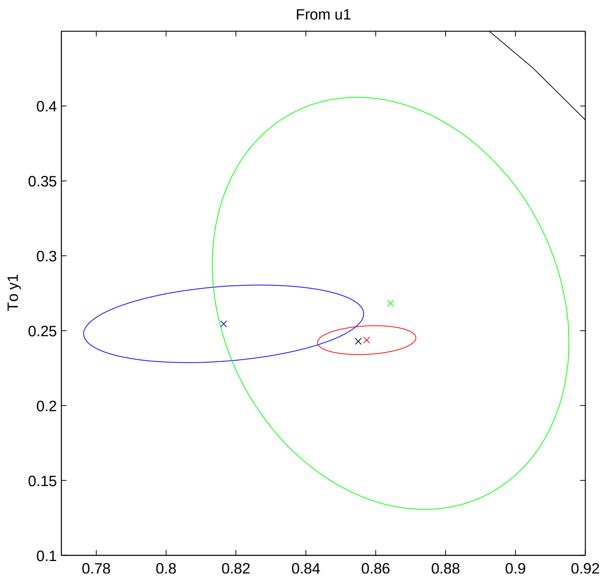


Only amplitude curves shown here.

A bandwidth of 3 rad/s gives an approximate scan rate of 3 rad/s. It is then important that we have good knowledge of the system around this frequency. In the frequency response, it is clearly seen that `z2` gives a extremely uncertain model above 2 rad/s (which is reasonable since we are not exciting the system there!). Of the other two data sets, `z1` gives a slightly more uncertain model in this area, which is explained by lower input signal energy. Therefore, `z3` is most suitable to use for system identification.

The uncertainty can also be studied for poles and zeros (`Zeros and poles`) with confidence intervals:

```
figure; pzmap(G0,'k',m1,'b',m2,'g',m3,'r','sd',3);
```



One of the poles zoomed in.

(6p)

4. (a) For the change in volume of the liquid we have $\dot{V} = F_1 + F_2 - F_3$. The resulting heat transfer in the tank from the water flows is $q_{\text{flow}} = \rho c(F_1 T_1 + F_2 T_2 - F_3 T)$, unit Watt. The heat flow out through the tank wall is given by $q_{\text{out}} = AU(T - T_0)$. The relation $T = \frac{1}{C}\int q = \frac{1}{\rho V c}\int q$ is differentiated to obtain the power balance $\frac{d}{dt}(\rho V c T) = q$. This in turn is rewritten as $\rho c(\dot{V}T + V\dot{T}) = q$. The total heat transfer q is given by $q = q_{\text{flow}} - q_{\text{out}}$. This finally gives the relationships

$$\begin{aligned} \dot{V} &= F_1 + F_2 - F_3, \qquad V(0) = V_0 \\ \dot{T} &= \frac{1}{V}\Big(F_1(T_1 - T) + F_2(T_2 - T) - \frac{AU}{\rho c}(T - T_0)\Big) \end{aligned}$$

(7p)

(b) Dimensional analysis of $\dot{V}$ gives:
LHS: $[\dot{V}] = \frac{L^3}{T}$.
RHS: $[F_1, F_2, F_3] = \frac{L^3}{T}$. OK!
Dimensional analysis of $\dot{T}$ gives:
LHS: $[\dot{T}] = \frac{K}{T}$.
RHS: $[\frac{F_i T_i}{V}, \frac{F_i T}{V}] = \frac{L^3}{T} K \frac{1}{L^3} = \frac{K}{T}, \quad i = 1, 2, 3.$
$[\frac{AUT}{\rho c V}, \frac{AUT_0}{\rho c V}] = L^2 \frac{ML^2T^{-3}}{L^2K} K \frac{1}{ML^{-3}} \frac{1}{ML^2T^{-2}M^{-1}K^{-1}} \frac{1}{L^3} = \frac{K}{T}$. OK! (3p)