

Solution for the exam in the course “Modeling and Learning for Dynamical Systems” (TSRT92) 2022-10-28

1. (a) An algorithm for the solution of differential equations can be written as

$$x_{n+1} = G(t, x_{n-k+1}, x_{n-k+2}, \dots, x_n, x_{n+1}).$$

If x_{n+1} is not included in the expression, x_{n+1} can be obtained directly by evaluating the right-hand side, i.e. $x_{n+1} = G(t, x_{n-k+1}, x_{n-k+2}, \dots, x_n)$. For an implicit method, the solution x_{n+1} is in the right-hand side and a system of equations must then be solved to obtain x_{n+1} .

An implicit method gives a higher order of accuracy than an explicit method with the same stability region, but is more computationally demanding. (2p)

- (b) Conduct several step response experiments with different amplitudes. A linear system should give twice the output signal at twice the magnitude of the step as input signal. (2p)

- (c) Case (i):

$$\begin{aligned} y_2(t) &= 0.1y_2(t-1) + u_2(t) = /u_2(t) = y_1(t)/ \\ &= 0.1y_2(t-1) - 0.4y_1(t-1) + 0.2y_1(t-2) + u_1(t) = /u_1(t) = y_2(t)/ \\ &= 0.1y_2(t-1) - 0.4y_1(t-1) + 0.2y_1(t-2) + y_2(t) \end{aligned}$$

- Case (ii):

$$\begin{aligned} y_2(t) &= u_2(t-1) + u_2(t-2) = /u_2(t) = y_1(t)/ \\ &= -0.4y_1(t-2) + 0.2y_1(t-3) + u_1(t-1) - 0.4y_1(t-3) + 0.2y_1(t-4) + u_1(t-2) \\ /u_1(t) &= y_2(t)/ \\ &= -0.4y_1(t-2) - 0.2y_1(t-3) + 0.2y_1(t-4) + y_2(t-1) + y_2(t-2) \end{aligned}$$

In (i), a term $y_2(t)$ appears in the right-hand side, which gives an algebraic loop. In (ii) it works well, since the time indices of y_2 in the right-hand side are shifted at least one step compared to the left-hand side. (2p)

- (d) Aggregate the state variables! This means that several variables of the same nature are combined into *one* state. The object can be divided into e.g. N different segments where the state of a particular segment becomes the homogeneous temperature in that segment. The temperature is thus assumed to be constant in the whole segment. (2p)

- (e) A stiff differential equation has solutions with both slow and fast components and the difference between these time constants is large. The fast term gives a requirement for a very short step length. Some methods (explicit) make it impossible to increase the step length gradually without getting stability problems in the solver. These stability problems can be avoided by using solvers that are always stable for $Re(\lambda h) < 0$, but then a low order of accuracy is obtained. Solution methods for stiff problems therefore become a compromise between a large stability region (almost the entire left half-plane) and a reasonably high order of accuracy. (2p)

2. (a) The system is written on the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}$$

The fast and slow dynamics of the system are obtained by

$$\det(\lambda I - A) = \begin{bmatrix} \lambda + 1 & 0 & 0 \\ 0 & \lambda + 15 & 0 \\ -1 & 0 & \lambda + 2 \end{bmatrix} = 0$$

which gives $\lambda_1 = -1, \lambda_2 = -15, \lambda_3 = -2$. The fast dynamics is thus given by the state x_2 . Replace the dynamics for x_2 , i.e. $\dot{x}_2 = -15x_2 + 10u$, with the static relation given by $\dot{x}_2 = 0 = -15x_2 + 10u$. This gives $x_2 = \frac{2}{3}u$. The approximate system is now given by

$$\begin{aligned}\dot{x}_1 &= -x_1 + u \\ \dot{x}_3 &= x_1 - 2x_3 + 2u \\ y &= 3x_1 + 0.5 \cdot \frac{2}{3}u + x_3 = 3x_1 + x_3 + \frac{1}{3}u\end{aligned}\tag{3p}$$

- (b) i. Force equilibrium for the ball gives $m\ddot{r}(t) = -\frac{kI(t)}{r^2(t)} + mg$.
Choosing $x_1(t) = r(t)$, $x_2(t) = \dot{r}(t)$ as the state, we obtain

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{k}{mx_1^2(t)}I(t) + g \\ y(t) &= x_1(t)\end{aligned}\tag{3p}$$

- ii. The stationary point is given by $\dot{x} = 0$. This gives $x_2 = 0$ and

$$0 = -\frac{kI_0}{x_1^2} + mg \iff x_1 = \pm\sqrt{\frac{kI_0}{mg}}$$

The negative solution can be rejected since the ball is under the magnet all the time, i.e. $r = x_1(t) > 0$. (1p)

- iii. With the notation $\dot{x} = f(x, I)$, the linearization can be written as

$$\Delta\dot{x} = \frac{\partial f(x, I)}{\partial x}\Delta x + \frac{\partial f(x, I)}{\partial I}\Delta I = \begin{bmatrix} 0 & 1 \\ \frac{2kI_0}{mx_1^3} & 0 \end{bmatrix}\Delta x + \begin{bmatrix} 0 \\ -\frac{k}{mx_1^2} \end{bmatrix}\Delta I$$

where Δx and ΔI represent the deviation from respective stationary value and x_1 is the stationary value that was given in task ii. The poles are given by

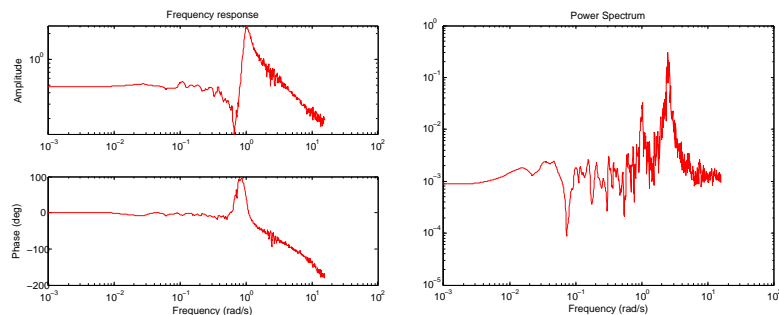
$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 \\ -\frac{2kI_0}{mx_1^3} & \lambda \end{bmatrix} = 0$$

which gives the poles $\lambda = \pm\sqrt{\frac{2kI_0}{mx_1^3}}$. (3p)

3. (a) Start by subtracting the average of the input and output signal. The reason for this is that the linear black-box models we study do not explicitly estimate the absolute signal levels, which is why it is unnecessary to put effort into getting the estimate to the correct mean level. (2p)
- (b) Remove averages and split into estimation and validation data as below. Then a spectral estimate is produced, where we used 1000 frequencies, logarithmically distributed. (This gives slightly less flutter at high frequencies than with linear distribution.) (2p)

```
z = iddata(y,u,0.2); % iddata-object
zd = detrend(z,0); % remove means and trends
ze = zd([1:length(y)/2]);
zv = zd(length(y)/2+1:length(y));
spafdr1000 = spafdr(ze, [], logspace(log10(0.001), log10(15.708), 1000));
```

It can be seen that the system probably has a notch in the frequency response at about 0.63 rad/s and a peak at about 1 rad/s. When compared to a lower number of frequencies during the spectral estimation, it is seen that the upper of the two peaks of the disturbance spectrum becomes more and more prominent with 1000 frequencies, suggesting that the upper peak at about 2.5 rad/s is the true disturbance. This is true with what we know about the system (the disturbances contain mainly a relatively high frequency component). Considering the data spectrum, one also sees that there is quite a lot of output energy at resonance peak of the system at 1 rad/s, and a small part of it is also included when estimating the noise model, as we just noted. (2p)



- (c) When selecting the model order (at the “knee” of the loss function, **Order Selection**), one sees that order 3 gives reasonably good selection with a fit of 47.02 %. For comparison, the “best” ARX model proposed by **Order Selection** is also taken, of order 10, which only gives a fit of 54.19 %. An appropriate choice is to stay at a low order, since higher order still does not give much better results.

The OE models suggest choosing order 3. Order 2 gives 50.23 % alignment, while order 3 gives 74.17 % alignment. Orders above this give very similar results to order 3 and are therefore not recommended as they do not add anything.

As for the ARMAX models, it can be seen that order 5 gives the best result with an adaptation of 74.27 %. As we will see in the c) task, the fit of orders below

this is due to the fact that not the whole system and the noise can be estimated correctly. Higher order than 5 does not give better results due to overfitting.

ARX331, OE331 and AMX5551 are the models selected for further comparisons.

```

arx331 = arx(ze, [3 3 1]);
oe331 = oe(ze, [3 3 1]);
amx5551 = armax(ze, [5 5 5 1]);

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(3p)

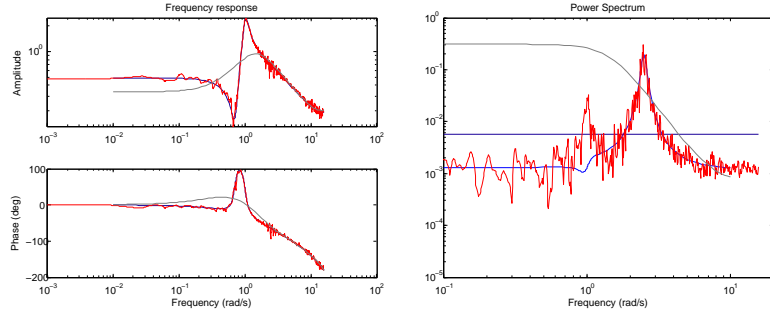
- (d) An ARX model of order higher than 4 gives pole-zero cancellation, so it may be overfitted to the noise. It is also seen that no ARX model is capable of estimating the system frequency response satisfactorily, nor the noise. This is not surprising, since the ARX model has the same poles for both the system and the noise model, which is not the case here. I.e. an ARX model is a bad choice here.

The OE331 can estimate the frequency function of the system, but as you know it does not have a noise model.

The AMX5551 is the best choice of them all, as it can estimate both the system and the noise. However, it suffers from some pole-zero cancellation. It can also be noted that the OE331 and AMX5551 have very similar poles and zeros, i.e. the OE331 does well except for the noise model that the AMX5551 adds.

In the figure, the fuzzy line is from SPAFDR with 1000 frequencies. ARX331 is the mismatched curve, while OE331 and AMX5551 follow the SPAFDR curve well for the system (left figure). For the noise model (right figure), it is seen that neither ARX331 (bent curve) nor OE331 (straight curve) follow the SPAFDR model, while the AMX5551 does.

(3p)



4. (a) We start by defining the dimensions of the included variables (where we denote the dimensions of mass, length and time with M , L and T):

$$\begin{aligned}
[p] &= [p_0] = ML^{-1}T^{-2} \\
[r] &= L \\
[m] &= M \\
[\rho] &= ML^{-3} \\
[E] &= [V \frac{\partial p_{\text{sur}}}{\partial V}] = ML^{-1}T^{-2}.
\end{aligned}$$

We are looking for an expression of p given as a function of the other variables, i.e.,

$$p = f(p_0, r, m, \rho, E).$$

According to Buckingham's theorem, such a relationship should be possible to write as a relationship between dimensionless variables. The next step is therefore to figure out which dimensionless variables can be formed out of the included variables, i.e., we are looking for an expression having the form

$$p^{e_1} \cdot p_0^{e_2} \cdot r^{e_3} \cdot m^{e_4} \cdot \rho^{e_5} \cdot E^{e_6}.$$

The dimension of such a variable becomes

$$\begin{aligned} & [p^{e_1} \cdot p_0^{e_2} \cdot r^{e_3} \cdot m^{e_4} \cdot \rho^{e_5} \cdot E^{e_6}] \\ &= (ML^{-1}T^{-2})^{e_1} (ML^{-1}T^{-2})^{e_2} L^{e_3} M^{e_4} (ML^{-3})^{e_5} (ML^{-1}T^{-2})^{e_6} \\ &= M^{e_1+e_2+e_4+e_5+e_6} L^{-e_1-e_2+e_3-3e_5-e_6} T^{-2e_1-2e_2-2e_6}. \end{aligned}$$

In order for it to be dimensionless, the exponents need to fulfill

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 1 & 0 & -3 & -1 \\ -2 & -2 & 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_6 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This is an equation system with 6 unknowns and 3 linearly independent equations, and thus $6 - 3 = 3$ linearly independent solutions. One way to select them:

$$e^{(1)} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad e^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad e^{(3)} = \begin{bmatrix} 0 \\ 0 \\ -3 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

which corresponds to the dimensionless variables

$$\frac{p}{p_0}, \quad \frac{p_0}{E}, \quad \frac{m}{\rho r^3},$$

(we could have seen the 2 first variables instantly without doing any calculations).
(7p)

- (b) A relationship for the 6 original variables will according to Buckingham's theorem assume the form

$$F\left(\frac{p}{p_0}, \frac{p_0}{E}, \frac{m}{\rho r^3}\right) = 0,$$

or, if we solve for p ,

$$p = p_0 g\left(\frac{p_0}{E}, \frac{m}{\rho r^3}\right).$$

(3p)