Solution for the exam in the course "Modeling and Learning for Dynamical Systems" (TSRT92) 2022-08-22

1. a) The predictor of a NARX model is, in general, given by

$$\hat{y}(t|\theta) = g\big(y(t-1), \dots, y(t-n), u(t), \dots, u(t-m), \theta\big)$$

where g is a non-linear function. Writing it as neural network gives e.g.

$$\hat{y}(t|\theta) = \sum_{k=1}^{d} \alpha_k \kappa(\beta_k^T(\varphi - \gamma_k))$$

with $\theta = \{\alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_d, \gamma_1, \dots, \gamma_d\}$. $\kappa(x)$ is a so-called base function, which is a non-linear scaling function of a scalar variable x.

The difference is that g ia a non-linear function instead of a linear function as in a normal ARX. That is,

$$\hat{y}(t) = -a_1 y(t-1) + \dots + -a_n y(t-n) + b_1 u(t) + \dots + b_m u(t-m)$$

b) Assume that a step with amplitude u_0 gives a step response $y_0(t)$. Then, steps in the input signal with amplitudes $u_k = \alpha_k u_0$, $k = 1, \dots, d$ will give output signals $y_k(t) = \alpha_k y_0(t)$, $k = 1, \dots, d$. If not, the system is non-linear. Note: Even if the test above is valid, the system *can* be non-linear of other types of input signals or of other amplitudes of the input signal than the ones tested.

c) Since u changes slowly, the parts with fast time constants can be replaced with static relationships. For the system

$$\dot{x}_1 = -350x_1 + 70u \qquad (1a)$$
$$\dot{x}_2 = 10x_1 - 3x_2 + u \qquad (1b)$$
$$y = 5x_1 + 4x_2 \qquad (1c)$$

is (1a) much faster than (1b). Therefore, (1a) can be replaced with the static relation

$$0 = -5x_1 + u$$

This gives the system

$$\dot{x}_2 = -3x_2 + 3u$$
$$y = 4x_2 + u$$

d) Introduce the scaling

$$\tau = \beta t, \quad z(t) = \alpha y(\beta t) = \alpha y(\tau)$$

The derivation of z is then

$$\frac{dz(t)}{dt} = \alpha \frac{dy(\tau)}{d\tau}\frac{d\tau}{dt} = \alpha\beta \frac{dz(\tau)}{d\tau}, \quad \frac{d^2 z(t)}{dt^2} = \alpha\beta^2 \frac{d^2 y(\tau)}{d\tau^2}$$

If we plug in these expressions in

$$\frac{d^2z(t)}{dt^2} + 8\frac{dz(t)}{dt} + 12z(t) = u(t)$$

we get

$$\alpha\beta^2\frac{d^2y(\tau)}{d\tau^2} + 8\alpha\beta\frac{dy(\tau)}{d\tau} + 12\alpha y(\tau) = u(\tau/\beta)$$

By choosing the parameters β and α as

$$\alpha = 1/4, \quad \beta = 2$$

we get the equation for which you have the solution. In other words, in order to get the solution to the desired differential equation, the time axis must be compressed by a factor of 2 and the amplitude reduced to 1/4 (ty $z(t) = 1/4y(2t)$).

e) The spectrum of y is given by

$$\Phi_y(\omega) = |G(i\omega)|^2\Phi_u(\omega) + \Phi_e(\omega) = G(i\omega)G(-i\omega)\Phi_u(\omega) + \Phi_e(\omega) = \frac{1}{1+\omega^2} + 1$$

since

$$\Phi_u(\omega) = \Phi_e(\omega) = 1$$

2. (a) The estimate $\hat{\theta}_N$ will converge to θ^* according to

$$\theta^* = \lim_{N\to\infty} \hat{\theta}_N = \arg\min_{\theta} \int_{-\pi}^{\pi} \mid G_0(e^{i\omega}) - G(e^{i\omega}, \theta) \mid^2 \Phi_u(\omega) d\omega$$

where $G_0(e^{i\omega})$ is the true system, $G(e^{i\omega}, \theta)$ is the model and $\Phi_u(\omega)$ is the signal spectrum. (The model of the controller is $H_*(e^{i\omega}) = 1$ here.) Thus, the model convergence is weighted with the input signal spectrum $\Phi_u(\omega)$.

Since we have the right model structure, there are values θ_0 such that $G_0(e^{i\omega}) = G(e^{i\omega}, \theta_0)$. Since the input signal is white noise, is $\Phi_u(\omega)$ constant and the result above gives that $\theta^* = \theta_0$, that is $\hat{a}_1 = -1.71$, $\hat{a}_2 = 0.79$, $\hat{b}_1 = 1$ och $\hat{b}_2 = 0.92$.

(b) Due to the constant input signal, the parameter convergence will be focused entirely on $\omega = 0$ because $\Phi_u(\omega) = 0$ for $\omega \neq 0$. That is,

$$\hat{b} = \arg\min_{b} \mid G_0(e^{i0}) - b \mid^2 = \frac{1 + 0.92}{1 - 1.71 + 0.79} \approx 24$$

(c) We start by loading the dataset and studying its time and frequency characteristics (can also be done in the user interface by selecting `Time plot` or `Data spectra`):

```
load ex091012_4c
figure; plot(z1,z2,z3)
figure; plot(fft(z1),fft(z2),fft(z3))
```

We immediately see that the maximum amplitudes of the input signals are the same, but that the spectrum differs. `z1` contains all frequencies between 0–31 rad/s, `z2` approximately 0–2 rad/s and `z3` approximately 1.5–7 rad/s . The spectrum of `z1` is also significantly lower because the energy is distributed over more frequencies.

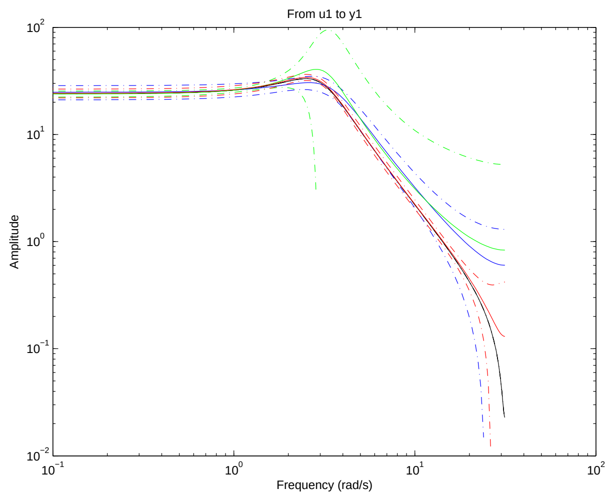
The models are estimated with:

```
m1=oe(z1,[2 2 1]);
m2=oe(z2,[2 2 1]);
m3=oe(z3,[2 2 1]);
```

If you look at the coefficients, it is above all b_1 and b_2 that varies a lot, but even the poles differ.

The models are evaluated above all in the frequency plane (`Frequency resp`) with confidence intervals plotted:

```
G0=idpoly(1,[0 1 0.92],1,1,[1 -1.71 0.79],[],0.1); % the true system
figure; bode(G0,'k',m1,'b',m2,'g',m3,'r','sd',3); % 3 standard deviations
```

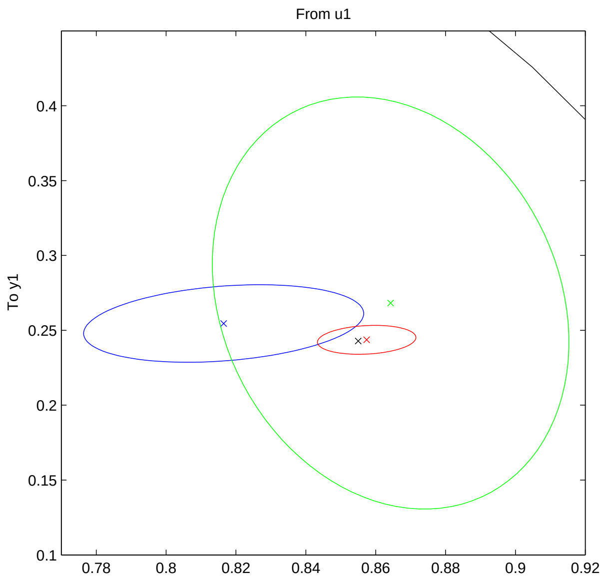


Only amplitude curves shown here.

A bandwidth of 3 rad/s gives an approximate scan rate of 3 rad/s. It is then important that we have good knowledge of the system around this frequency. In the frequency response, it is clearly seen that `z2` gives a extremely uncertain model above 2 rad/s (which is reasonable since we are not exciting the system there!). Of the other two data sets, `z1` gives a slightly more uncertain model in this area, which is explained by lower input signal energy. `z3` is therefore satisfying.

The uncertainty can also be studied for poles and zeros (`Zeros and poles`) with confidence intervals:

```
figure; pzmap(G0,'k',m1,'b',m2,'g',m3,'r','sd',3);
```



One of the poles zoomed in.

3. a) Backward approximation gives

$$\begin{pmatrix} \frac{1}{h}(x_{1,n} - x_{1,n-1}) - x_{2,n} \\ x_{1,n} - t_n^2 \end{pmatrix} = 0$$

which gives

$$x_{1,n} = t_n^2$$
$$x_{2,n} = \tfrac{1}{h}(x_{1,n} - x_{1,n-1}) = \tfrac{1}{h}(t_n^2 - x_{1,n-1})$$

Here, the local and the global error are the same since $x_{1,n-1} = x_1(t_{n-1})$, that is

$$x_1(t_n) - x_{1,n} = t_n^2 - t_n^2 = 0$$
$$x_2(t_n) - x_{2,n} = 2t_n - \frac{1}{h}(t_n^2 - t_{n-1}^2) = 2t_n - \frac{1}{h}(t_n^2 - (t_n - h)^2) = h$$

That is, the error is h.

b) Let $n(k)$ be the number of tins at time k. Then,

$$n(k+1) = n(k) + u_{in}(k) - u_{ut}(k)$$

Let the average age at time k be called $x(k)$. Consider the change between two times k and $k+1$. At time k , $u_{in}(k)$ tins are added while $u_{out}(k)$ are removed. At the next time (i.e. $k+1$), the tins that were added have become one day old while the amount of tins that remained, i.e. $n(k) - u_{ut}(k)$, have become one day older. That gives the equation

$$x(k+1) = \frac{u_{in}(k) + \big(n(k) - u_{ut}(k)\big)\big(x(k) + 1\big)}{n(k) + u_{in}(k) - u_{ut}(k)}$$

where the model is only valid if $n(k) > 0$ and $x(k) \geq 0$. A plausibility test of the model above is that $u_{in} = 0$ gives $x(k+1) = x(k)+1$, i.e. if no new tins are added, the entire stock gets older at the same rate that time does.

Stationary points are obtained by estimating $n(k) = n(k+1) = \bar{n}$ and $x(k) = x(k+1) = \bar{x}$. This gives the equations

$$\begin{aligned} 0 &= u_{in,0} - u_{ut,0} \\ 0 &= \bar{n} - \bar{x} u_{ut,0} \end{aligned}$$

where $u_{in,0}$ and $u_{ut,0}$ are constant input signals. This means that in order for a stationary point to exist, the input signals must be chosen to be equal and the stationary point is then obtained as

$$(\bar{n},\ \bar{x}) = \left(\bar{n},\ \frac{\bar{n}}{u_{ut,0}}\right)$$

i.e. it is parameterized in both $u_{ut,0}$ and $\bar{n}$. Even for the stationary points, a small plausibility check can be made. If the turnover of tins is large in relation to the number, i.e. if $u_{out,0}$ and thus also $u_{in,0}$, is large in relation to $\bar{n}$, a low average age is obtained.

Assume instead that you take out the oldest tins from the warehouse. Let $\bar{n}_i$ be the number of tins of age i at the stationary point and let d be the age of the oldest tins. Then, the model is obtained by

$$\begin{aligned} \bar{n}_1 &= u_{in,0} \\ \bar{n}_2 &= \bar{n}_1 \\ &\vdots \\ 0 &= \bar{n}_d - u_{ut,0} \end{aligned}$$

where the zero in the last row is due to att no tins should be older, which also is obtained since $\bar{n}_i = u_{in,0}$, $i = 1, \ldots, d$. The average age becomes

$$\bar{x} = \frac{\sum_{i=1}^{d} i u_{in,0}}{d u_{in,0}} = \frac{1+d}{2}$$

4. (a) Write the system in matrix form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u$$

with $x = [x_1 \quad x_2 \quad x_3]^T$ och $u = [u_1 \quad u_2]^T$. Now differentiate the equations until the matrix in front of $\dot{x}$ gets full rank. The number of derivatives of the system is the index of the system. Taking the derivative of row 3 gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \dot{u}$$

The matrix in front of $\dot{x}$ has still rank 2. Add row 1 to row 3 and subtract row 2:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix} x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \dot{u}$$

Taking the derivative of row 3 gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & -2 \end{bmatrix} \dot{x} + \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \dot{u} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \ddot{u}$$

and the matrix in front of $\dot{x}$ has full rank. Since we have differentiated two times, the index is two.

Answer: Index 2. Motivation see above.

(b) With $u_2 = l_1 x_1 + l_2 x_2 + l_3 x_3$ the matrix form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -(1+l_1) & (1-l_2) & -l_3 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1$$

is obtained. Taking the derivative of the third row gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -(1+l_1) & (1-l_2) & -l_3 \end{bmatrix} \dot{x} + \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1$$

The requirement to obtain index 1 is that the matrix in front of $\dot{x}$ must have rank 3, which is the case when $l_3 \neq 0$ and l_1 and l_2 arbitrary.

Answer: The system has index 1 when $l_3 \neq 0$, l_1 and l_2 arbitrary.