# Dynamical systems and Control

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# Lecture 9: State feedback

- PID issues
- State feedback

# PID issues

# PID controller

- Is not always the best choice

- Does not always work

# Example: Water level regulation

$x_i$: Water level

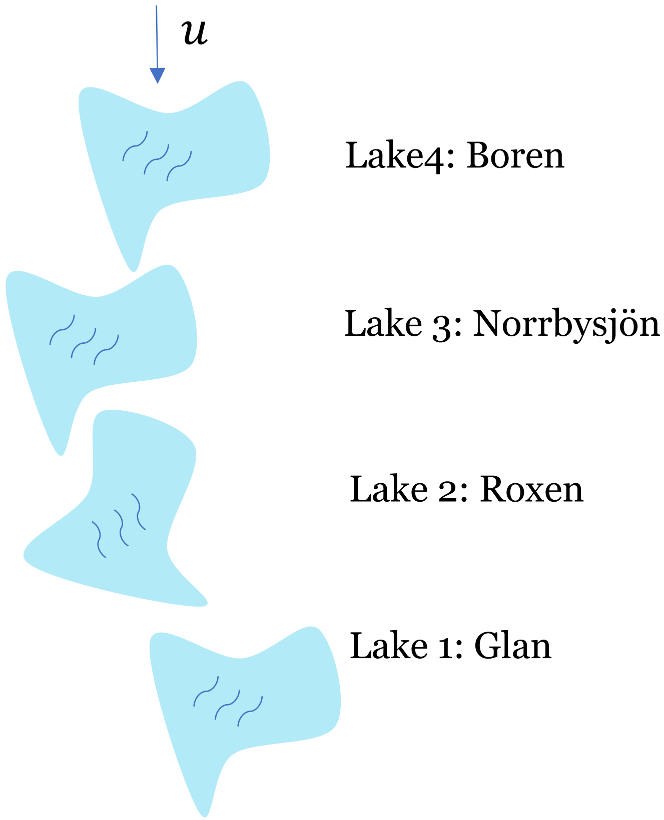
$$\dot{x}_4 = -\alpha_4 x_4 + u$$

$$\dot{x}_3 = -\alpha_3 x_3 + x_4$$

$$\dot{x}_2 = -\alpha_2 x_2 + x_3$$

$$\dot{x}_1 = -\alpha_1 x_1 + x_2$$

$$y = x_1$$



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$$\dot{x}_2 = -\alpha_2 x_2 + x_3$$

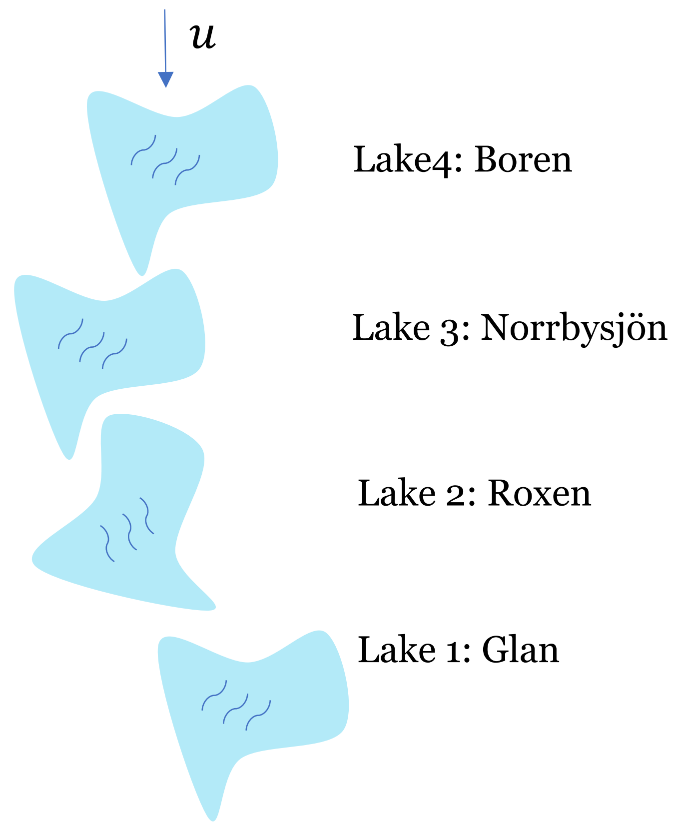
$$\dot{x}_1 = -\alpha_1 x_1 + x_2$$

$$y = x_1$$

A PD controller:

$$u = K_P(r - y) + K_D(\dot{r} - \dot{y})$$

$$u = K_P(r - x_1) + K_D(\alpha_1 x_1 - x_2)$$



# Example: Vehicle platooning

$d_i$: Distance to the front vehicle

$v_i$: Velocity of vehicle $i$

# State feedback

# State-space representation

Recap from session 1

$$\dot{x} = Ax + Bu$$

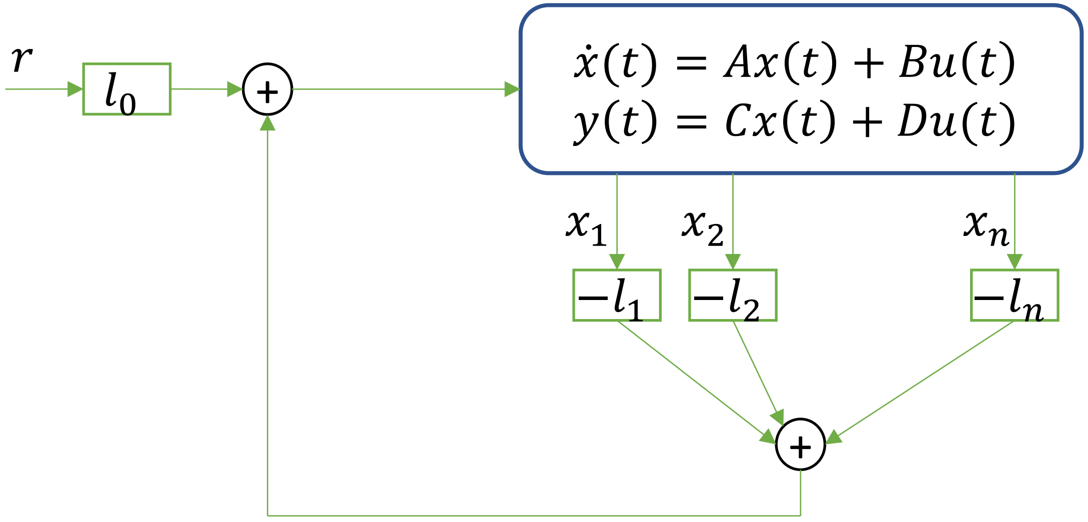
$$y = Cx + Du$$

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Discussion: what is $x$?

# Linear state feedback

$$u(t) = -l_1 x_1(t) - l_2 x_2(t) \cdots - l_n x_n(t) + l_0 r(t)$$




$$u(t) = -Lx(t) + l_0 r(t)$$

# Closed loop system

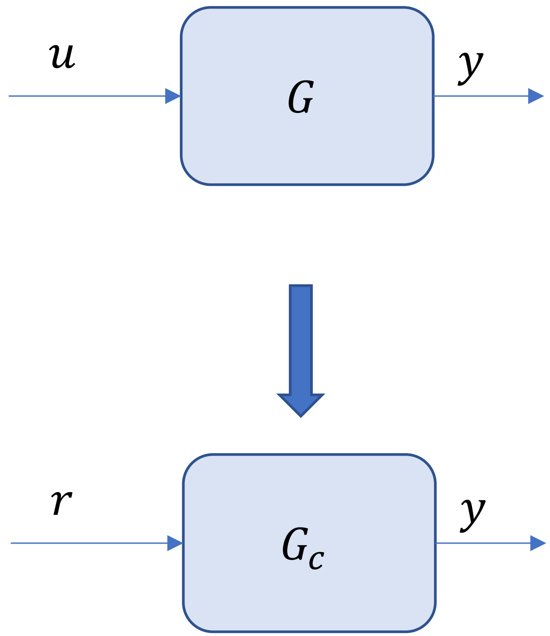
$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$u = -Lx + l_0 r$$

$$\dot{x} = (A - BL)x + Bl_0 \mathrm{r}$$

$$y = (C - DL)x + Dl_0 \mathrm{r}$$




Quiz: Derive $G_c$!

# Closed-loop system

The new poles are given by

$$\det\big(\lambda I - (A - BL)\big) = 0$$

**Design procedure:**

1. Select the desired poles
2. Design $L$ to have the desired poles
3. Select $l_0$ to have a zero tracking-error for a step $r$

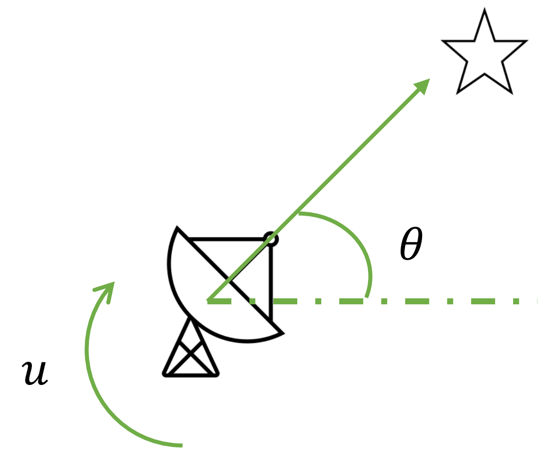
# Example- Angle control on satellite

Control the viewing angle $\theta$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$$x_1 = \theta, \qquad x_2 = \dot{\theta}$$



# Example- continued

$$u = -l_1 x_1 - l_2 x_2 + l_0 r$$

# Example- continued

$$u = -l_1 x_1 - l_2 x_2 + l_0 r = -[l_1 \quad l_2] x + l_0 r$$

The closed-loop system reads

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} u = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x - \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} [l_1 \quad l_2] x + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} l_0 r$$

$$= \begin{bmatrix} 0 & 1 \\ -0.01 l_1 & -0.01 l_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} l_0 r$$

# Example- continued

1- Select the desired poles

2- Design $L$ to have the desired poles

# Example- continued

1- Select the desired poles at $\lambda_{1,2} = -0.1, -0.2$

$$\lambda^2 + 0.3\lambda + 0.02 = 0$$

2- Design $L$ to have the desired poles

$$\det\big(\lambda I - (A - BL)\big) = 0$$

$$\lambda I - (A - BL) = \begin{bmatrix} 0 & 1 \\ -0.01l_1 & -0.01l_2 \end{bmatrix}$$

Take $det$

$$\lambda^2 + 0.01l_2\lambda + 0.01l_1 = 0$$

# Example- continued

# Example- continued

$$\lambda^2 + 0.01 l_2 \lambda + 0.01 l_1 = 0 \qquad \equiv \qquad \lambda^2 + 0.3\lambda + 0.02 = 0$$

$$l_2 = 30$$

$$l_1 = 2$$

Use place(A,B,[-0.1,-0.2])

Thanks MATALB!

# Example- continued

3. Select $l_0$ to have a zero tracking-error for a step $r$

Approach 1:

# Example- continued

3. Select $l_0$ to have a zero tracking-error for a step $r$

Approach 1: Set $G_c(0) = 1$

Final value theorem: $y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sG_c(s)\frac{1}{s} = \lim_{s\to 0} G_c(s)$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -0.02 & -0.3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} l_0 r$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$$G_c(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -0.02 & -0.3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0.01 l_0 \end{bmatrix} = \frac{1}{s^2+0.3s+0.02} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+0.3 & 1 \\ -0.02 & s \end{bmatrix} \begin{bmatrix} 0 \\ 0.01 l_0 \end{bmatrix}$$

$$G_c(s) = \frac{0.01 l_0}{s^2 + 0.3s + 0.02} \longrightarrow l_0 = 2$$

# Example- continued

3. Select $l_0$ to have a zero-tracking error for a step $r$

Approach 2:

# Example- continued

3. Select $l_0$ to have a zero-tracking error for a step $r$

Approach 2: Study the stationary point

$\dot{x} = 0$ → The system is in a stationary point

$y = r$ → Perfect tracking

$$0 = \begin{bmatrix} 0 & 1 \\ -0.02 & -0.3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} l_0 1$$

$1 = [1 \quad 0]x$ → $x_1 = 1$

Substituting in the first equation:

$x_2 = 0$

$-0.2 + 0.01 l_0 = 0$ → $l_0 = 2$

# Example- continued

The final controller

$$u = -2x_1 - 30x_2 + 2r$$

Discussion: Any relations to PID?

# What do we cover next?

- PID and state feedback

# Ask us!

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