# Dynamical systems and Control

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# Lecture 3: State-space representation

- Recap
- Transfer function of state-space model
- Controllability and observability
- Linearization

# Characteristics of first and second order systems

A quick recap of lecture 2

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# First-order system

$$T\dot{y}(t) + y(t) = Ku(t)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{Ts + 1}$$

- $T$: Time constant
- $K$: Static gain
- The pole is $\lambda = -\frac{1}{T}$

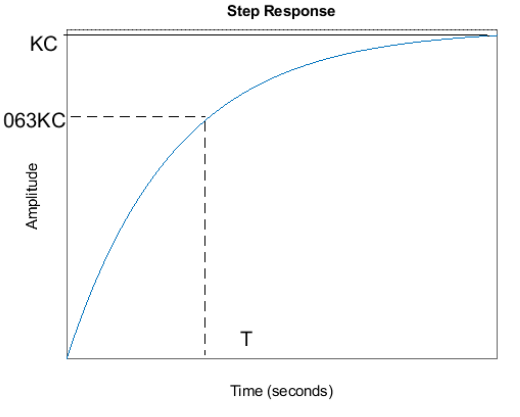
# First-order system

$$T\dot{y}(t) + y(t) = Ku(t)$$

Take Laplace:

$$TsY(s) + Y(s) = KU(s) \longrightarrow G(s) = \frac{Y(s)}{U(s)} = \frac{K}{Ts+1}$$

- $T$: Time constant- The time the output reaches 63% of its final value
- $K$: Static gain: How much the input is magnified in the steady state
- The pole is $\lambda = -\frac{1}{T}$



# Second-order system

$$G(s) = \frac{a_2}{s^2 + a_1 s + a_2}$$

Complex poles

$$G(s) = K\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \longrightarrow \lambda_{1,2} = \omega_0(-\zeta \pm i\sqrt{1-\zeta^2})$$

# Example

$$G(s) = \frac{4}{s^2 + 5s + 9}$$

# Example

$$G(s) = \frac{4}{s^2 + 5s + 9}$$

$$G(s) = \frac{4}{9}\frac{3^2}{s^2 + 2 \times \frac{5}{6} \times 3s + 3^2}$$

Compare with

$$G(s) = K\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

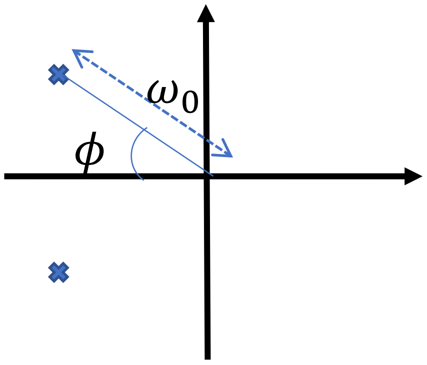
Then, we have

$$K = \frac{4}{9} \qquad \zeta = \frac{5}{6} \qquad \omega_0 = 3$$

# Second-order system with complex poles

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

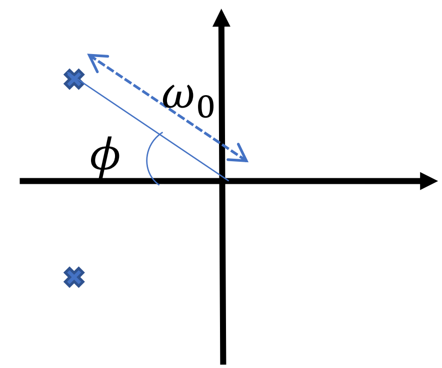
Complex poles: $\lambda_{1,2} = \omega_0(-\zeta \pm i\sqrt{1-\zeta^2})$



# Second-order system with complex poles

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Complex poles: $\lambda_{1,2} = \omega_0(-\zeta \pm i\sqrt{1-\zeta^2})$




Write the pole in polar system:

$\omega_0$ = Distance to the origin $\sqrt{\omega_0^2\zeta^2 + \omega_0^2(1-\zeta^2)} = \omega_0$

$\zeta = \cos\phi$ relative damping $\cos\phi = \frac{\omega_0\zeta}{\omega_0} = \zeta$

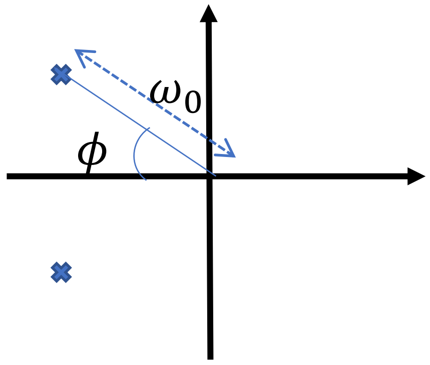
# Second-order system with complex poles

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Complex poles: $\lambda_{1,2} = \omega_0(-\zeta \pm i\sqrt{1-\zeta^2})$

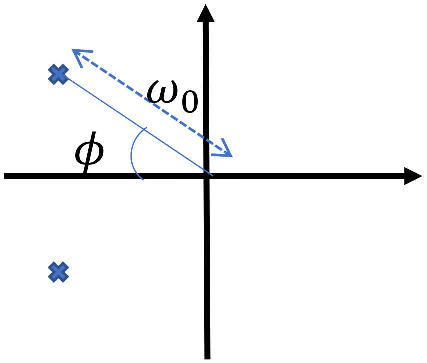
Step response:

$$y(t) = 1 - \frac{e^{-\zeta\omega_0 t}}{\sqrt{1-\zeta^2}} \sin(\omega_0\sqrt{1-\zeta^2}t + \phi)$$



# Second-order system with complex poles

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$




Complex poles: $\lambda_{1,2} = \omega_0(-\zeta \pm i\sqrt{1-\zeta^2})$

Step response:

$$Y(s) = \frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}\frac{1}{s} = \frac{1}{s} - \frac{s+2\zeta\omega_0}{s^2+2\zeta\omega_0 s+\omega_0^2}$$

Take inverse Laplace

$$y(t) = 1 - \mathcal{L}^{-1}\left\{\frac{s + 2\zeta\omega_0}{s^2 + 2\zeta\omega_0 s + \omega_0^2}\right\}$$

Use A.27 on p. 233, let $a = \zeta\omega_0$, $b = \omega_0\sqrt{1-\zeta^2}$, $z = 2\zeta\omega_0$

$$y(t) = 1 - \frac{e^{-\zeta\omega_0 t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos\omega_0\sqrt{1-\zeta^2}t + \zeta\sin\omega_0\sqrt{1-\zeta^2}t\right)$$

Introduce $\zeta = \cos\phi$, $\omega_d = \omega_0\sqrt{1-\zeta^2}$

$$y(t) = 1 - \frac{e^{-\zeta\omega_0 t}}{\sqrt{1-\zeta^2}}(\sin\phi\cos\omega_d t + \cos\phi\sin\omega_d t) = 1 - \frac{e^{-\zeta\omega_0 t}}{\sqrt{1-\zeta^2}}\sin(\omega_d + \phi)$$

# Second-order system with complex poles

$$y(t) = 1 - \frac{e^{-\zeta\omega_0 t}}{\sqrt{1-\zeta^2}} \sin(\omega_0 \sqrt{1-\zeta^2} t + \phi)$$

$\zeta$: relative damping:

$\omega_0$: kind of time scaling

# Second-order system with complex poles

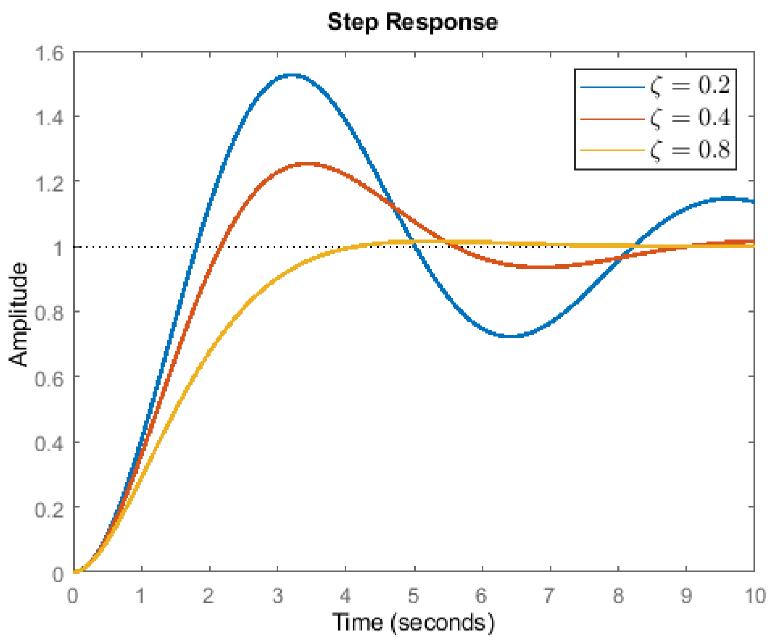
$$y(t) = 1 - \frac{e^{-\zeta\omega_0 t}}{\sqrt{1-\zeta^2}} \sin(\omega_0 \sqrt{1-\zeta^2} t + \phi)$$

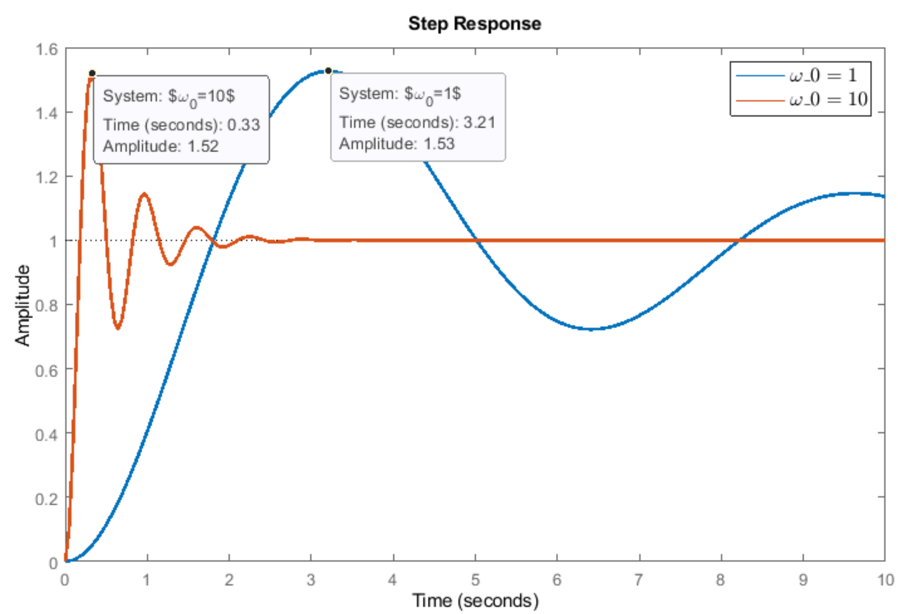
$\zeta$: relative damping:

- **Small:** weak damping (more oscillations)
- **Big:** strong damping (less oscillations)

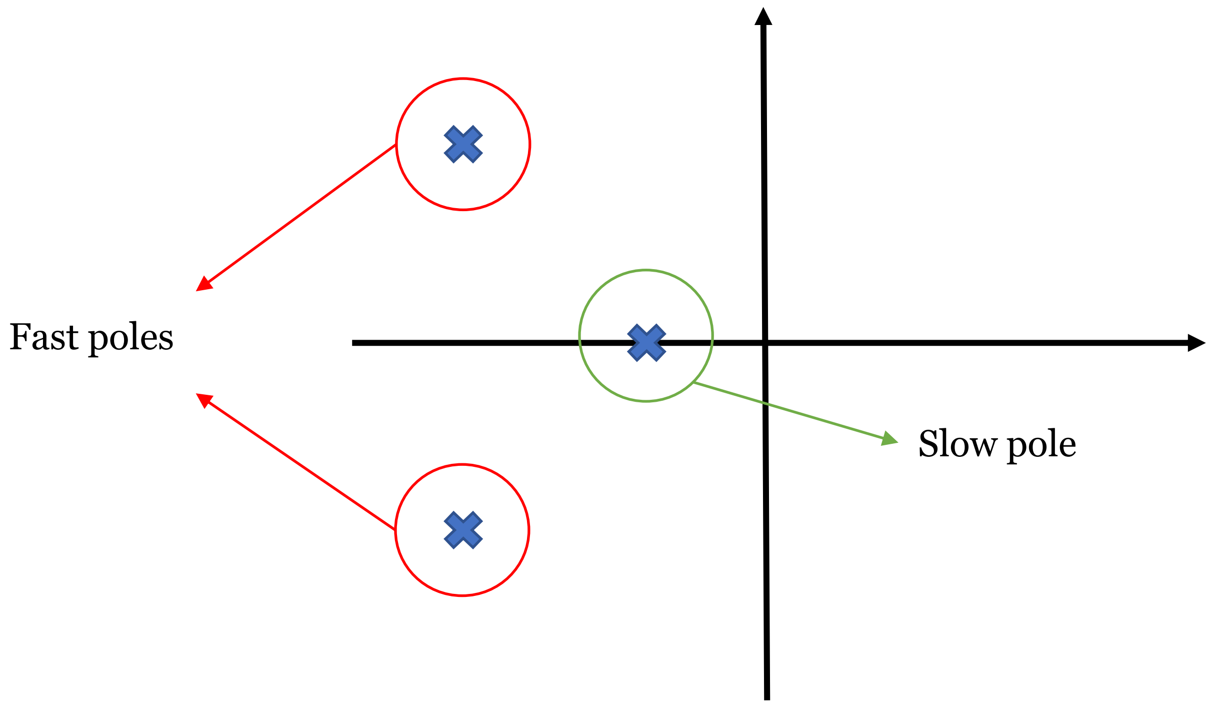
$\omega_0$: kind of time scaling

- E.g. 10 times bigger, 10 times faster



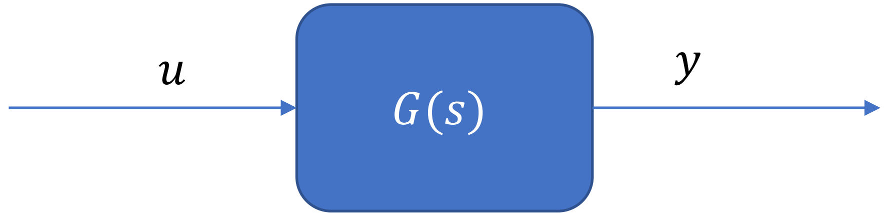



# Fast and slow dynamics



# Transfer function of state-space model

# State-space model

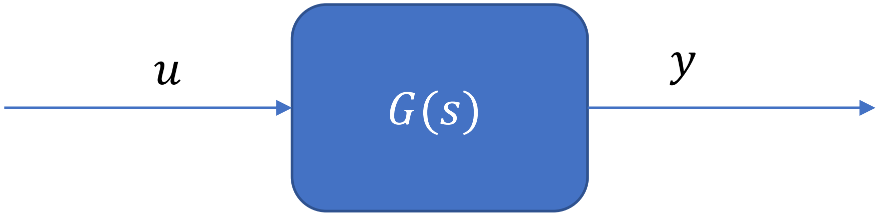



$$\dot{x} = Ax + Bu \qquad (1)$$

$$y = Cx + Du \qquad (2)$$

$$G(s) = C(sI - A)^{-1}B + D$$

# State-space model




$$\dot{x} = Ax + Bu \qquad (1)$$

$$y = Cx + Du \qquad (2)$$

First, note that

$$\mathcal{L}\{\dot{x}\} = \begin{bmatrix} \mathcal{L}\{\dot{x}_1\} \\ \vdots \\ \mathcal{L}\{\dot{x}_n\} \end{bmatrix} = \begin{bmatrix} sX_1(s) \\ \vdots \\ sX_n(s) \end{bmatrix} = \begin{bmatrix} s & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & s \end{bmatrix} = sI_nX(s)$$

We take the Laplace transformation of (1)

$$sI_nX(s) = AX(s) + BU(s) \rightarrow X(s) = (sI - A)^{-1}BU(s)$$

Now, we replace this result in Laplace of (2)

$$Y(s) = CX(s) + DU(s) = (C(sI - A)^{-1}B + D)U(s)$$

$$G(s) = C(sI - A)^{-1}B + D$$

# Example- What is the transfer function for Boeing 747

$\alpha$: Angle of attack

$q$: Pitch rotation rate

$$\begin{bmatrix}\dot{\alpha}\\ \dot{q}\end{bmatrix} = \begin{bmatrix}-0.39 & 1\\ -1.55 & -0.53\end{bmatrix}\begin{bmatrix}\alpha\\ q\end{bmatrix} + \begin{bmatrix}-18.5\\ -1.2\end{bmatrix}\delta_e$$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\alpha\\ q\end{bmatrix}$$

**Solution:**

# Example- What is the transfer function for Boeing 747

$$\begin{bmatrix}\dot{\alpha}\\ \dot{q}\end{bmatrix} = \begin{bmatrix}-0.39 & 1\\ -1.55 & -0.53\end{bmatrix}\begin{bmatrix}\alpha\\ q\end{bmatrix} + \begin{bmatrix}-18.5\\ -1.2\end{bmatrix}\delta_e$$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\alpha\\ q\end{bmatrix}$$

Solution:

$$(sI - A)^{-1} = \begin{bmatrix}s+0.39 & -1\\ 1.55 & s+0.53\end{bmatrix}^{-1} = \frac{1}{(s+0.39)(s+0.53)+1.55}\begin{bmatrix}s+0.53 & 1\\ -1.55 & s+0.39\end{bmatrix}$$

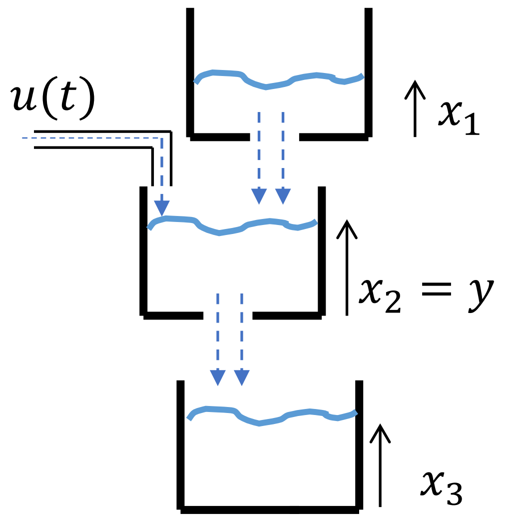
$$G(s) = \frac{1}{(s+0.39)(s+0.53)+1.55}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}s+0.53 & 1\\ -1.55 & s+0.39\end{bmatrix}\begin{bmatrix}-18.5\\ -1.2\end{bmatrix}$$

$$G(s) = \frac{-1.2s + 28.2}{s^2 + 0.92s + 1.7567}$$

Use $ss2tf(A, B, C, D)$ in MATLAB!

# Controllability and Observability

# Example- A triple-tank



# Controllability

**Definition:** The system is controllable if there exists an input signal that brings the state of the system to any desired $x^*$ from the origin.

**Method:** The system is controllable if

$$rank([B\ AB\ \ \dots\ A^{n-1}B] = n)$$

# Observability

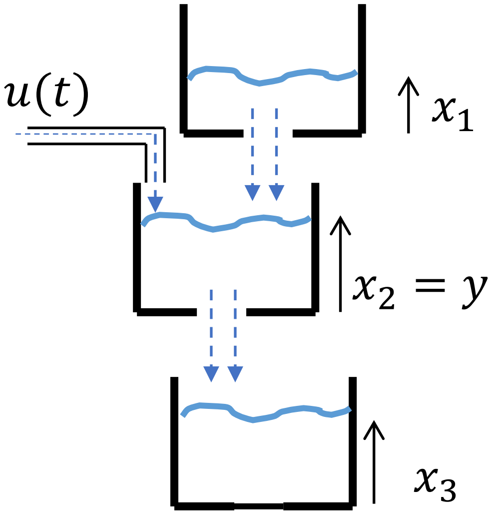
**Definition:** Let $u = 0$. If $x^*(0) \neq 0$ leads to $y(t) = 0$, we say that $x^*$ is unobservable. If there is no unobservable state, then the system is observable

**Method:** The system is observable if

$$rank(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n)$$

# Example- Is the triple-tank controllable and observable?

**Solution:**



# Example- Is the triple-tank controllable and observable?

**Solution:** write differential equations

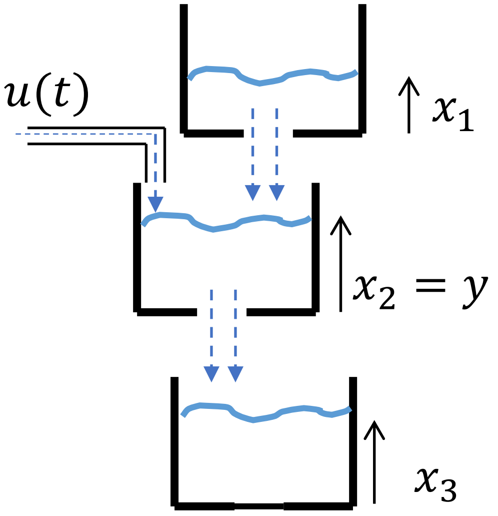
$\dot{x}_1 = -x_1$

$\dot{x}_2 = x_1 - x_2 + u \qquad y = x_2$

$\dot{x}_3 = x_2$

Now, write them in state space format:

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x$$




Controllability:

$$[B, AB, A^2B] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \rightarrow Det([B, AB, A^2B]) = 0$$

Not controllable

Observability:

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -2 & 1 & 0 \end{bmatrix} \rightarrow Det\left(\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}\right) = 0$$

Not observable

Transfer function: $G = \dfrac{1}{s+1}$

Two poles are missing!

# Linearization

# Nonlinear model

$$\dot{x}(t) = f(x(t), u(t))$$

$$y(t) = h(x(t), u(t))$$

Stationary point:

$$f(x_0, u_0) = 0$$

$$y_0 = h(x_0, u_0)$$

# Linearized model

Let

$$\Delta x = x - x_0$$
$$\Delta u = u - u_0$$
$$\Delta y = y - y_0$$

Linearized model:

$$\dot{\Delta x} = A\Delta x + B\Delta u \qquad A = \frac{\partial f}{\partial x}\Big|_{x=x_0, u=u_0} \qquad B = \frac{\partial f}{\partial u}\Big|_{x=x_0, u=u_0}$$

$$\Delta y = C\Delta x + D\Delta u \qquad C = \frac{\partial h}{\partial x}\Big|_{x=x_0, u=u_0} \qquad D = \frac{\partial h}{\partial u}\Big|_{x=x_0, u=u_0}$$

# Linearized model

Let

$$\Delta x = x - x_0$$
$$\Delta u = u - u_0$$
$$\Delta y = y - y_0$$

Derive $\dot{\Delta x}$ and use Taylor expansion

$$\dot{\Delta x} = \dot{x} - \dot{x}_0 = f(x,u) = f(x_0 + \Delta x, u_0 + \Delta u) \approx f(x_0, u_0) + \frac{\partial f}{\partial x}|_{x=x_0,u=u_0}\Delta x + \frac{\partial f}{\partial u}|_{x=x_0,u=u_0}\Delta u$$
$$\approx \frac{\partial f}{\partial x}|_{x=x_0,u=u_0}\Delta x + \frac{\partial f}{\partial u}|_{x=x_0,u=u_0}\Delta u$$

Write $\Delta y$

$$\Delta y = h(x,u) - y_0 = h(x_0 + \Delta x, u_0 + \Delta u) - y_0 \approx h(x_0, u_0) + \frac{\partial h}{\partial x}|_{x=x_0,u=u_0}\Delta x + \frac{\partial h}{\partial u}|_{x=x_0,u=u_0}\Delta u - y_0$$
$$\approx \frac{\partial h}{\partial x}|_{x=x_0,u=u_0}\Delta x + \frac{\partial h}{\partial u}|_{x=x_0,u=u_0}\Delta u$$

Linearized model:

$$\dot{\Delta x} = A\Delta x + B\Delta u \qquad A = \frac{\partial f}{\partial x}|_{x=x_0,u=u_0} \qquad B = \frac{\partial f}{\partial u}|_{x=x_0,u=u_0}$$
$$\Delta y = C\Delta x + D\Delta u \qquad C = \frac{\partial h}{\partial x}|_{x=x_0,u=u_0} \qquad D = \frac{\partial h}{\partial u}|_{x=x_0,u=u_0}$$

# Linearized model

$n$ states, $m$ inputs, $p$ outputs

$$A = \frac{\partial f}{\partial x}\Big|_{x=x_0,u=u_0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \qquad B = \frac{\partial f}{\partial u}\Big|_{x=x_0,u=u_0} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}$$

$$C = \frac{\partial h}{\partial x}\Big|_{x=x_0,u=u_0} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial x_1} & \cdots & \frac{\partial h_p}{\partial x_n} \end{bmatrix} \qquad D = \frac{\partial h}{\partial u}\Big|_{x=x_0,u=u_0} = \begin{bmatrix} \frac{\partial h_1}{\partial u_1} & \cdots & \frac{\partial h_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial u_1} & \cdots & \frac{\partial h_p}{\partial u_m} \end{bmatrix}$$

# Linearized model

$n$ states, $m$ inputs, $p$ outputs

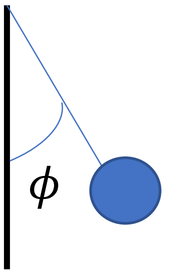
$$f(x,u) = \begin{bmatrix} f_1(x,u) \\ \vdots \\ f_n(x,u) \end{bmatrix} \qquad h(x,u) = \begin{bmatrix} h_1(x,u) \\ \vdots \\ h_p(x,u) \end{bmatrix}$$

$$A = \frac{\partial f}{\partial x}|_{x=x_0,u=u_0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \qquad B = \frac{\partial f}{\partial u}|_{x=x_0,u=u_0} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}$$

$$C = \frac{\partial h}{\partial x}|_{x=x_0,u=u_0} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial x_1} & \cdots & \frac{\partial h_p}{\partial x_n} \end{bmatrix} \qquad D = \frac{\partial h}{\partial u}|_{x=x_0,u=u_0} = \begin{bmatrix} \frac{\partial h_1}{\partial u_1} & \cdots & \frac{\partial h_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial u_1} & \cdots & \frac{\partial h_p}{\partial u_m} \end{bmatrix}$$

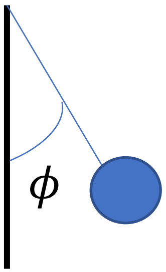
# Example- Linearize the model

$J\ddot{\phi} = -mg\sin\phi - c\dot{\phi}$




**Solution:**

# Example- Linearize the model




$$J\ddot{\phi} = -mg\sin\phi - c\dot{\phi}$$

**Solution:** Let

$$x_1 = \phi \qquad x_2 = \dot{\phi}$$

Write the space state equations:

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -\frac{mg}{J}\sin x_1 - \frac{c}{J}x_2$$

Stationary point by $\dot{x}_1 = 0, \dot{x}_2 = 0$:
case1) $x_{20} = 0, x_{10} = 0$ or
case 2) $x_{20} = 0, x_{10} = \pi$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{mg}{J}\cos x_1 & -\frac{c}{J} \end{bmatrix}$$

case1) $x_{20} = 0, x_{10} = 0$ $\quad A_1 = \frac{\partial f}{\partial x}|_{x=x_0,} = \begin{bmatrix} 0 & 1 \\ -\frac{mg}{J} & -\frac{c}{J} \end{bmatrix}$ stable

case2) $x_{20} = 0, x_{10} = \pi$ $\quad A_1 = \frac{\partial f}{\partial x}|_{x=x_0,} = \begin{bmatrix} 0 & 1 \\ \frac{mg}{J} & -\frac{c}{J} \end{bmatrix}$ unstable

# What do we cover next?

- Sensor
- Introduction to signal processing

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# Ask us!

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