Solutions to the exam on Dynamical systems and Control, TSRT21, on 22 March 2022

1. The answer to question 1:
   (a) The reference r is the desired volume of coffee by the user, the output is the volume of the coffee, and the input u is the fluid which is changed by the valve.
   (b) A P controller cannot eliminate the permanent error. One can use a PI controller. The integrator term can eliminate the error.
2. The answer to question 2:
   (a) Using Laplace transform
   $$s^2Y + 3sY + 2 = -3sU + U. \tag{1}$$
   So the transfer function is $G(s) = \frac{Y}{U} = \frac{-3s+1}{s^2+3s+2}$.
   (b) The poles are given by the roots of the denominator: $p_{1,2} = -1, -2$. The zero is given by the root of the numerator: $z_1 = \frac{1}{3}$.
   (c) The system is stable because the real parts of all poles are negative.
3. The answer to question 3:
   (a) $H_1(z)$: A high-pass filter, $H_2(z)$: A low-pass filter, $H_3(z)$: A band-pass filter,
   (b) One can use H_2 which is a low pass filter and can pass 0.1 rad/s and filters hight frequency noise or H_3 that also passes 0.1 rad/s and filters high frequencies.
4. The answer to question 4:
   (a) One has the following in the z domain
   $$zY_f - 0.8Y_f = 0.4Y.$$
   Using the inverse of z-transform, one gets the following difference equation
   $$y_f(k+1) - 0.8y_f(k) = 0.4y(k), \text{ or } y_f(k) - 0.8y_f(k-1) = 0.4y(k-1), \tag{2}$$
   According to the sampling Theorem $w_0 \leq \frac{\pi}{T_s} = \pi$.
   (b) The filter $\frac{1}{10}(z^{-1} + z^{-2} + ... + z^{-10})$ computes the average of the last 10 inputs. So the filter $F_2(z) = 1 - \frac{1}{10}(z^{-1} + z^{-2} + ... + z^{-10})$ removes the average of the input and is a high-pass filter.
5. The answer to question 5:

   Step 1) According to the location of the desired poles $p_{1,2} = -2 \pm i2$, the desired characteristic equation reads
   $$(\lambda - p_1)(\lambda - p_2) = \lambda^2 + 4\lambda + 8 = 0.$$
   Step 2) Select $u = -l_1x_1 - l_2x_2 + l_0r = -Lx + l_0r$. The closed-loop system reads
   $$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 1 - l_1 & -1 - l_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ l_0 \end{bmatrix} r, \\ y &= \begin{bmatrix} 2 & 0 \end{bmatrix} x. \end{aligned}$$
   The characteristic equation of the closed loop is given by
   $$Det(\lambda I - (A - BL)) = Det(\begin{bmatrix} \lambda & -1 \\ l_1 - 1 & \lambda + 1 + l_2 \end{bmatrix}) = \lambda^2 + \lambda(1 + l_2) + l_1 - 1.$$
   Equating the desired characteristic equation and the closed-loop characteristic equation, we have $l_1 = 9$, $l_2 = 3$.

   Step 3) Select l_0 to have a zero tracking error for a step reference. First, the transfer function for the closed-loop system reads
   $$\begin{aligned} G_c(s) =& C(sI - (A - BL))^{-1}B \\ =& \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 8 & s+4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ l_0 \end{bmatrix} = \frac{1}{s^2 + 4s + 8} \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} s+4 & 1 \\ -8 & s \end{bmatrix} \begin{bmatrix} 0 \\ l_0 \end{bmatrix} \\ =& \frac{1}{s^2 + 4s + 8} \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} l_0 \\ l_0 s \end{bmatrix} = \frac{2l_0}{s^2 + 4s + 8} \end{aligned}$$

To have zero tracking error, $y(\infty) = r(\infty) = 1$ (because the input is a step).

$$y(\infty) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sG_c(s)\frac{1}{s} = G_c(0) = \frac{2l_0}{8} = 1.$$

So, $l_0 = 4$ and the controller is $u = -9x_1 - 3x_2 + 4r$.

6. The answer to question 6:

   (a) $b = \frac{2}{3}$, $L = 1$. Then the PID controller is given by

   $$u_{PID}(t) = 1.8(e(t) + \frac{1}{2}\int_0^t e(\tau)d\tau + 0.5\dot{e}(t))$$

   (b) If the input to the system is bounded and the control effort is too big, the generated control signal cannot be applied to the system and it will be cut off. This results in the error being accumulated in the integrator part of the PID controller. There are many ways to avoid the integral windup. One possible solution is to stop integrating the error when the input to the system is saturated. Another possible solution is to adjust the integrator by adding or subtracting some value to avoid saturated input.