Solutions to the exam on Dynamical systems and Control, TSRT21, on 22 March 2022

1. The answer to question 1:

   (a) The reference r is the desired level $6m$, the output is the level of water in lake Boren, and the input u is the flow of water into Boren.

   (b) The P and I components can result in oscillations and sometimes instability. By introducing a D component, we can have less oscillations and a better stability profile.

2. The answer to question 2:

   (a) The poles are given by the roots of the denominator: $p_{1,2} = -2 \pm i\sqrt{5}$. The zero is given by the root of the numerator: $z_1 = -2$.

   (b) The system is stable because the real parts of all poles are negative.

   (c) Figure 1.(b) can be the step response of the system. The plots in 1.(c) and 1.(d) are unstable. The final value of the output for the system is given by

   $$y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sG(s)\frac{1}{s} = G(0) = \frac{2}{9}.$$

   So, 1.(b) is the answer.

3. The answer to question 3:

   (a) According to the sampling Theorem $w_0 \leq \frac{\pi}{T_s} = \pi$.

   (b) We take z-transformation

   $$Y_f(z)(1 - 0.95z^{-1}) = 0.05z^{-1}Y(z),$$
   $$H(z) = \frac{Y_f(z)}{Y(z)} = \frac{0.05}{z - 0.95}.$$

   (c) It is a low-pass filter. The magnitude is $0dB$ for low frequencies.

   (d) The filter can pass approximately $w \leq 0.1(rad/s)$. Any upper bound in the range $[0.05, 0.1]$ is acceptable.

4. The answer to question 4:

   (a) Let $e = r - y$. Then

   $$Y(s) = G(s)U(s) + V(s),$$
   $$U(s) = F(s)E(s) = F(s)(R(s) - Y(s)).$$

   Replacing the second equation in the first one

   $$Y(s) = G(s)F(s)(R(s) - Y(s)) + V(s).$$

   As a result

   $$Y(s) = \frac{G(s)F(s)}{1 + G(s)F(s)}R(s) + \frac{1}{1 + G(s)F(s)}V(s).$$

   (b) The closed-loop system reads

   $$G_c = \frac{F(s)G(s)}{1 + F(s)G(s)} = \frac{(K_P + K_D s)/(s^2 - 1)}{1 + (K_P + K_D s)/(s^2 - 1)}$$
   $$= \frac{K_P + K_D s}{s^2 x + K_D s + K_P - 1}.$$

   The closed-loop system is stable if $K_p > 1$ and $K_D > 0$.

5. The answer to question 5:

   Step 1) According to the location of the desired poles $p_{1,2} = -2 \pm i2$, the desired characteristic equation reads

   $$(\lambda - p_1)(\lambda - p_2) = \lambda^2 + 4\lambda + 8 = 0.$$

Step 2) Select $u = -l_1 x_1 - l_2 x_2 + l_0 r = -Lx + l_0 r$. The closed-loop system reads

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -l_1 & -1 - l_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ l_0 \end{bmatrix} r,$$
$$y = \begin{bmatrix} 4 & 0 \end{bmatrix} x.$$

The characteristic equation of the closed loop is given by

$$Det(\lambda I - (A - BL)) = Det(\begin{bmatrix} \lambda & -1 \\ l_1 & \lambda + 1 + l_2 \end{bmatrix}) = \lambda^2 + \lambda(1 + l_2) + l_1.$$

Equating the desired characteristic equation and the closed-loop characteristic equation, we have $l_1 = 8$, $l_2 = 3$.

Step 3) Select l_0 to have a zero tracking error for a step reference. First, the transfer function for the closed-loop system reads

$$\begin{aligned} G_c(s) =& C(sI - (A - BL))^{-1} B \\ =& \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 8 & s + 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ l_0 \end{bmatrix} = \frac{1}{s^2 + 4s + 8} \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} s + 4 & 1 \\ -8 & s \end{bmatrix} \begin{bmatrix} 0 \\ l_0 \end{bmatrix} \\ =& \frac{1}{s^2 + 4s + 8} \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} l_0 \\ l_0 s \end{bmatrix} = \frac{4 l_0}{s^2 + 4s + 8} \end{aligned}$$

To have zero tracking error, $y(\infty) = r(\infty) = 1$ (because the input is a step).

$$y(\infty) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sG_c(s)\frac{1}{s} = G_c(0) = \frac{4 l_0}{8} = 1.$$

So, $l_0 = 2$ and the controller is $u = -8x_1 - 3x_2 + 2r$.

6. The answer to question 6:

   (a) One can write the transfer function as $G(s) = \frac{0.75}{0.5s+1}$. So, $T = 0.5$, $K = 0.75$ and $y(\infty) = G(0) \times 2 = 1.5$.

   (b) The filter $\frac{1}{20}(z^{-1} + z^{-2} + ... + z^{-20})$ computes the average of the last 20 inputs. So the filter $H(z) = 1 - \frac{1}{20}(z^{-1} + z^{-2} + ... + z^{-20})$ removes the average of the input and is a high-pass filter.