Solutions TSRT19/23 2024-03-12

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1. (a) The goal is to maintain the pulse (output $y(t)$ measured using a pulse sensor) at a particular pace, such as 160 beats per minute (reference $r(t)$). One way to control the effort for the person on the treadmill is to increase and decrease the inclination with constant speed. An alternative way is to increase or decrease the speed of the conveyor belt with constant inclination. Let us assume we use the inclination θ as the input $u(t)$ and fix the speed at a constant value. A reasonable control strategy to start with as a first try would be to use an increasing inclination (up-hill) when the pulse is too low and down-hill when it is too high, i.e. $u(t) = K(r(t) - y(t))$. The biggest challenge in our control design is most likely the fact that it is very hard to model how a person reacts to the change in inclination, i.e. derive the model $Y(s) = G(s)U(s)$. Not only is it hard to derive this model for a particular person, but the model will vary massively between different persons. As an example, the time-constant for a well-trained athlete would be much larger than the time-constant for a unfit academic teacher. Hence, the controller has to be very robust, or different controllers will have to be derived for different levels of fitness. We could of course add integral and derivative action to improve performance.

   (b) Our first try could be to decrease the integral gain, since integral feedback is known to introduce oscillatory behavior. If this does not work, or if it leads to other side-effects, we might try to increase the derivative action. This has to be done with care though, since it easily amplifies the effect of measurement noise.

   (c) Controllable if $\det(B\,AB) \neq 0$ and observable if $\det\begin{pmatrix} C \\ CA \end{pmatrix} \neq 0$.

   $$\det(B\,AB) = \det\begin{pmatrix} 1 & -1 \\ 1 & -\alpha \end{pmatrix} = -\alpha + 1 \neq 0, \qquad \det\begin{pmatrix} C \\ CA \end{pmatrix} = \det\begin{pmatrix} 1 & \alpha \\ -1 & -\alpha^2 \end{pmatrix} = \alpha(\alpha - 1) \neq 0.$$

   Hence, if $\alpha \neq 0, 1$, the system is both controllable and observable. For $\alpha \neq 1$, the poles can be placed arbitrarily using state-feedback, and thus stabilizable. For $\alpha = 1$, the poles cannot be placed arbitrarily due to the loss of controllability. However, the system is already stable in this case, so it is already stabilized (using for example the trivial feedback $u = 0x$).

   (d) Since the system is linear, we know that a sinusoidal input of frequency ω leads to a sinusoidal output with frequency ω. Hence, if we write the input as $A\sin(\omega t + \phi)$ we immediately know that $\omega = 1$. A sinusoidal input is amplified with a gain $|G(i\omega)|$ and phase-shifted $\arg(G(i\omega))$. At $\omega = 1$ we have $|G(i1)| = \sqrt{2}$ and $\arg(G(i1) = -\pi/4$. We thus conclude that it has to hold that $\sin(t) = A\sqrt{2}sin(t + \phi - \pi/4)$. Hence, $A = 1/\sqrt{2}$ and $\phi = \pi/4$.

2. (a) We have $E(s) = R(s) - Y(s) = R(s) - G(s)(V(s) + F(s)E(s))$ so $E(s) = \frac{-G(s)}{1+G(s)F(s)}V(s) + \frac{G(s)F(s)}{1+G(s)F(s)}R(s)$ and we thus have that the sought transfer function is $\frac{-G(s)}{1+G(s)F(s)} = \frac{-1}{s(s+1)+K}$

   (b) The gain is given by $\left|\frac{1}{i\omega(i\omega+1)+K}\right| = \left|\frac{1}{K-\omega^2+i\omega}\right| = \frac{1}{\sqrt{(K-\omega^2)^2+\omega^2}}$. Since disturbance has amplitude 1 and we want error to be smaller in magnitude than 0.1, we must have gains $\frac{1}{\sqrt{(K-1^2)^2+1^2}} < 0.1$ and $\frac{1}{\sqrt{(K-5^2)^2+5^2}} < 0.1$, leading to $K > \sqrt{99} + 1$ and $K > \sqrt{75} + 25$ and consequently our condition is $K > \sqrt{75} + 25 \approx 33.66$.

   (c) A is faster than B. C is faster than D. A and B are oscillatory. C and D are non-oscillatory. I and II have resonance peaks. III and IV do not have resonance peaks. I has lower bandwidth than II. III has lower bandwidth than IV. Higher bandwidth leads to faster response. Resonance peak leads to oscillatory response. Hence $A - II$, $B - I$, $C - IV$, $D - III$.

3. (a) The two unstable poles move into the left half plane when $K > 50$. There is also a pole in the origin, but it moves into the left half plane immediately when $K > 0$.

   (b) For $K \leq 0.89$ no pole has left the real line.

(c) The starting points correspond to the poles of the open loop system $G(s)F(s)$. Since we starting point in the origin, it means we have a pure integrator in the open loop. Since the stationary error will be zero if we have a pole in the origin in the open loop system, we conclude that there will be no static error (if K is chosen such that the closed loop system is stable)

(d) There does not exist any such K since the initially unstable poles which move into the left half plane never become well damped (note scaling on the real vs imaginary axis).

(e) The number of start and end points (5 and 3) correspond to the number of poles and zeros in the open loop system. Assume $G(s) = \frac{A(s)}{B(s)}$ and $F(s) = K\frac{C(s)}{D(s)}$ where $C(s)$ is degree 1 and $D(s)$ is degree 2. The closed-loop system is given by $\frac{KA(s)C(s)}{B(s)D(s)+KA(s)C(s)}$. In root-loci we have analyzed a characteristic equation $P(s) + KQ(s) = 0$ which thus corresponds to $P(s) = B(s)D(s)$ and $Q(s) = A(s)C(s)$. Zeros of $P(s)$ correspond to start points and zeros of $Q(s)$ are end points. Since there are 5 start points and $D(s)$ is degree 2 we must have degree 3 in $B(s)$. Since there are 3 end points and $C(s)$ is degree 1 we must have degree 2 in $A(s)$.

4. (a) With the assumption that the true system is given by

$$G^0(s) = G(s)(1+\alpha)$$

it has to hold that the relative error, which does not affect the poles, is

$$\Delta G(s) = \alpha,$$

where $|\alpha| < 0.5$. Assuming $FG \to 0$ when $s \to \infty$, the robustness criteria says $F(s)$ has to be designed such that

$$|\frac{F(i\omega)G(i\omega)}{1+F(i\omega)G(i\omega)}| = |G_c(i\omega)| < 2,$$

Note that the robustness criteria only is a sufficient criteria, and thus can be very conservative for some cases.

(b) The amplitude margin is defined at the frequency ω_p where $\arg G_o(i\omega_p) = \arg G(i\omega_p)F(i\omega_p) = -180°$. Hence, $G_o(i\omega_p)$ is a real negative number. Let us call this $G_o(i\omega_p) = -\beta$ where $\beta > 0$. The amplitude margin is $\frac{1}{|G_o(i\omega_p)|}$ which thus is $\frac{1}{\beta}$. Note that $\beta < 1$ if the closed-loop system is stable.

The closed-loop system is defined as $G_c(s) = \frac{G_o(s)}{1+G_o(s)}$ so $|G_c(i\omega_p)| = \left|\frac{G_o(i\omega_p)}{1+G_o(i\omega_p)}\right| = \frac{|\beta|}{|1-\beta|} = \frac{\beta}{1-\beta}$. We can thus draw the conclusion

$$\frac{\beta}{1-\beta} < \gamma \Rightarrow \beta < \gamma - \gamma\beta \Rightarrow \frac{1+\gamma}{\gamma} < \frac{1}{\beta} = A_m$$

(c) We define the state as being the combination of the states in the electric motor and the gas engine. The total torque generated is $y(t) = y_1(t) + y_2(t) = C_1x_1 + C_2x_2 = \begin{pmatrix} C_1 & C_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. The state space model follows immediately

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u$$
$$y = \begin{pmatrix} C_1 & C_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

5. (a) One way is to Laplace transform the state space model, yielding $Y(s) = C(sI-A)^{-1}BU(s)$ $= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} s & -1 \\ 1 & s+1 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}U(s) = \frac{1}{s^2+s+1}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} s+1 & 1 \\ -1 & s \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}U(s) = \frac{1}{s^2+s+1}U(s)$.

Alternatively, the transfer function says $\ddot{y} + \dot{y} + y = u$, introduce the states $x_1 = y$ and $x_2 = \dot{y}$ and derive the state space equation and it turns out to coincide with Jenny´s realization.

(b) From the definition of $y(t)$ we have that $y(t) = x_1(t)$, and the first differential equation says $x_2(t) = \dot{x}_1(t)$, i.e., $x_2(t) = \dot{y}(t)$ which thus is the rotational acceleration.

(c) Design the state feedback controller $u = -Lx + l_0r$. The closed-loop pole polynomial is given by $det(sI-(A-BL)) = det\begin{pmatrix} s & -1 \\ 1+l_1 & s+1+l_2 \end{pmatrix} = s^2 + (1+l_2)s + 1 + l_1$. We place the poles in -2 thus aiming for pole polynomial s^2+4s+4 which is obtained with $L = \begin{pmatrix} 3 & 3 \end{pmatrix}$. The requirement on no stationary error for a step response tells us that the closed-loop static gain should 1, i.e. $G_C(0) = C(0I-(A-BL))^{-1}Bl_0 = 1$ which leads to $l_0 = 4$ since $C(0I-(A-BL))^{-1}B = 1/4$.