SENSOR FUSION

Particle Filter Properties

Sensor Fusion

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Standard Particle Filter

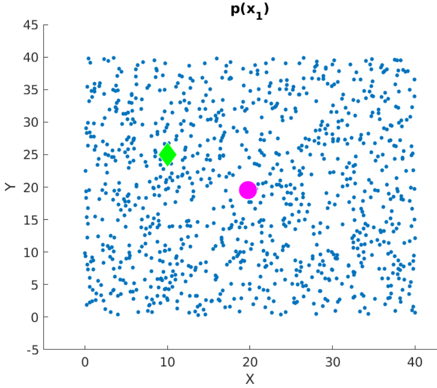
Choose N, generate $x_0^{(i)} \sim p_{x_0}, i = 1, \dots, N$,

1. *Measurement update:* For $k = 1, 2, \dots,$

$$\bar{w}_{k|k}^{(i)} = w_{k|k-1}^{(i)} p(y_k | x_k^{(i)}).$$

2. *Normalize:* $w_{k|k}^{(i)} := \bar{w}_{k|k}^{(i)} / \sum_j \bar{w}_{k|k}^{(j)}$.
3. *Estimation:* MMSE $\hat{x} \approx \sum_{i=1}^{N} w^{(i)} x^{(i)}$ or MAP.
4. *Resampling:* with replacement gives new set $\{x_k^{(i)}\}_{i=1}^{N}$ and $w_{k|k}^{(i)} = 1/N$.
5. *Prediction:* Generate random process noise samples

$$v_k^{(i)} \sim p_{v_k}, \quad x_{k+1}^{(i)} = f(x_k^{(i)}, v_k^{(i)}) \quad w_{k+1|k} = w_{k|k}.$$



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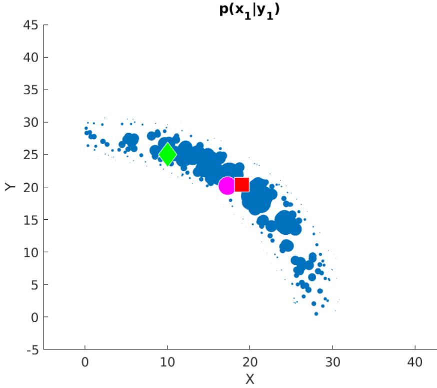

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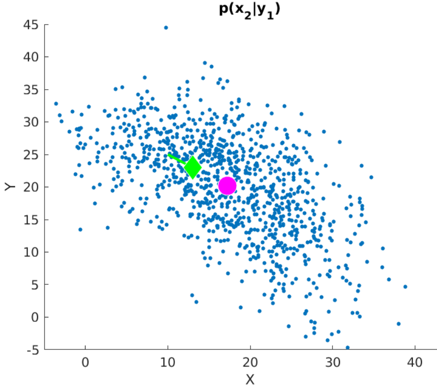

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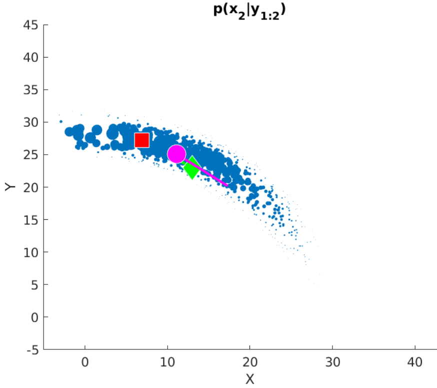

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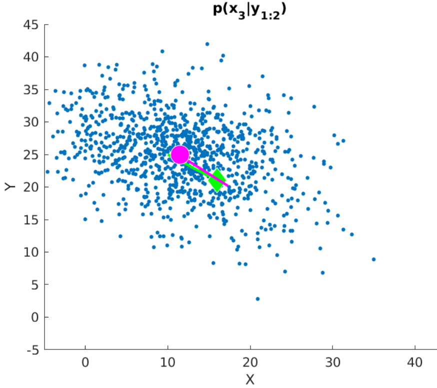

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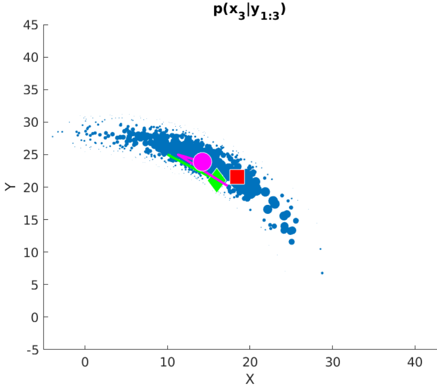

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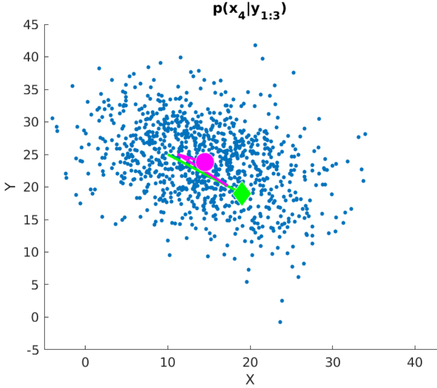

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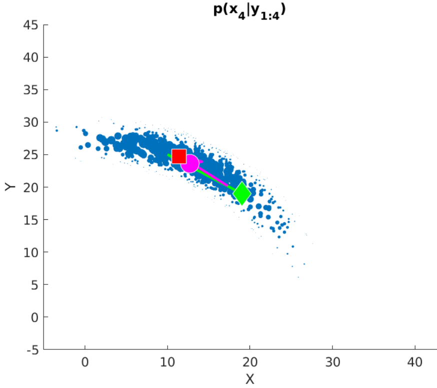

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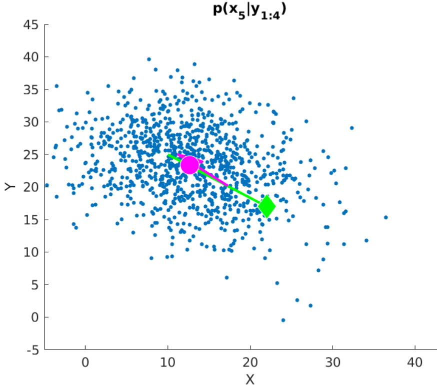

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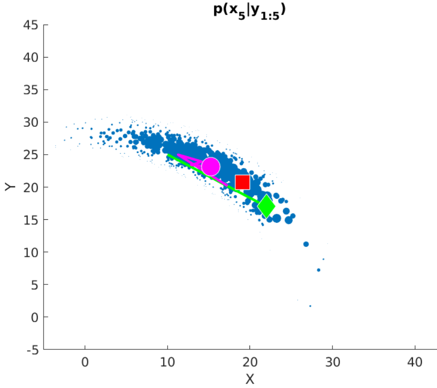

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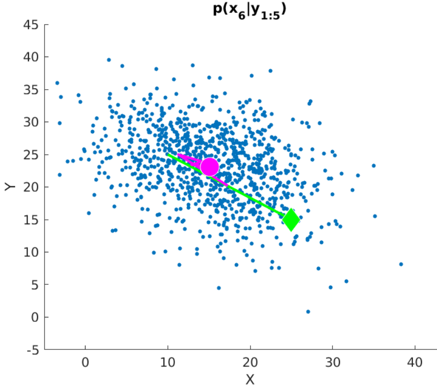

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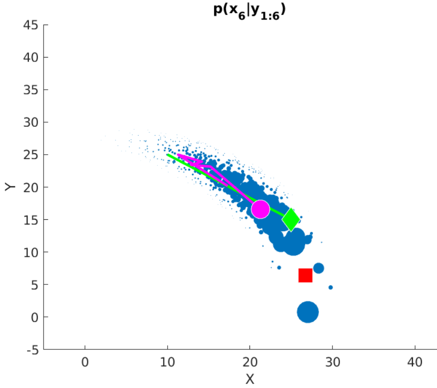

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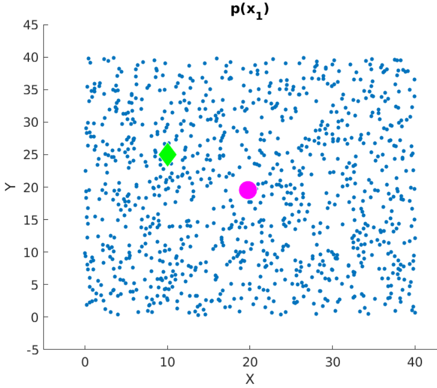

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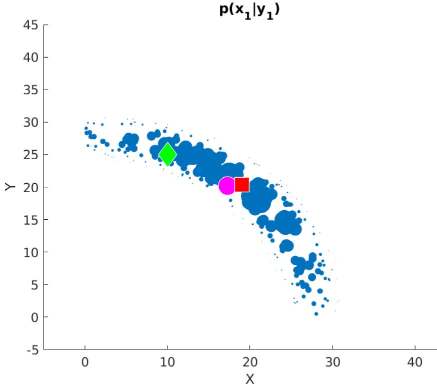

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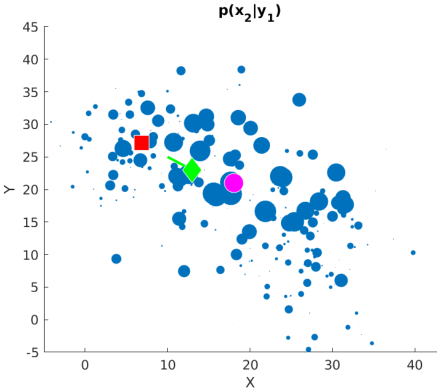

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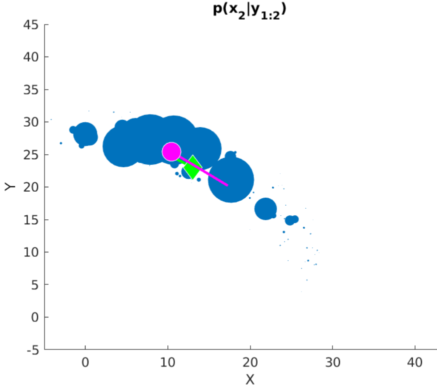

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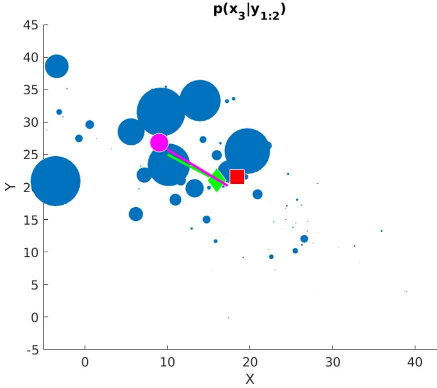

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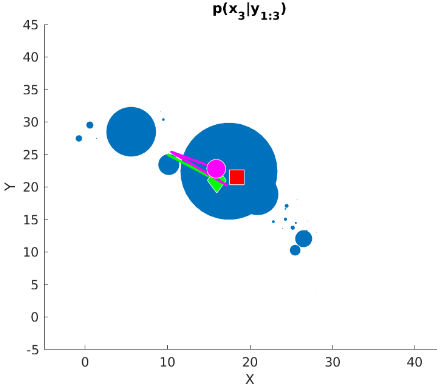

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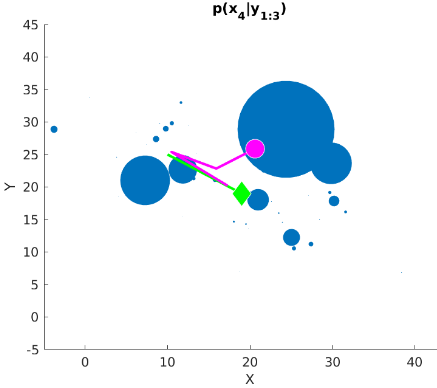

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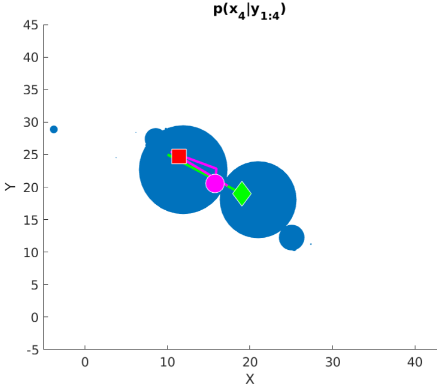
$$\bar{w}_{k|k}^{(i)} = w_{k|k-1}^{(i)} p(y_k | x_k^{(i)}).$$

2. *Normalize:* $w_{k|k}^{(i)} := \bar{w}_{k|k}^{(i)} / \sum_j \bar{w}_{k|k}^{(j)}$.
3. *Estimation:* MMSE $\hat{x} \approx \sum_{i=1}^{N} w^{(i)} x^{(i)}$ or MAP.
4. *Resampling:* with replacement gives new set $\{x_k^{(i)}\}_{i=1}^{N}$ and $w_{k|k}^{(i)} = 1/N$.
5. *Prediction:* Generate random process noise samples

$$v_k^{(i)} \sim p_{v_k}, \quad x_{k+1}^{(i)} = f(x_k^{(i)}, v_k^{(i)}) \quad w_{k+1|k} = w_{k|k}.$$



Green romb: ground truth
Magenta circle: estimate
Red square: measurement
Blue dots: particles

Standard Particle Filter (no resampling)

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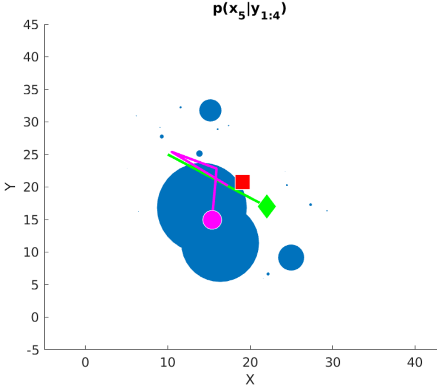

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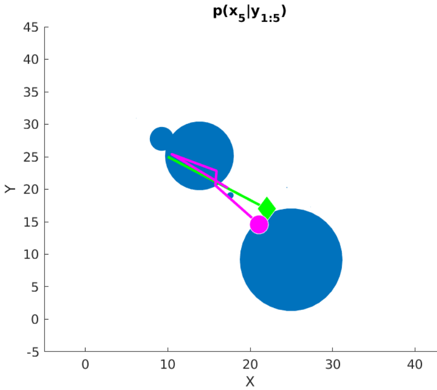

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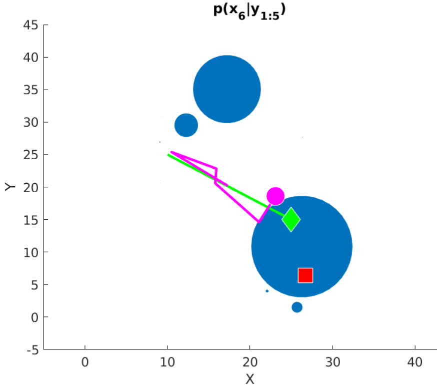

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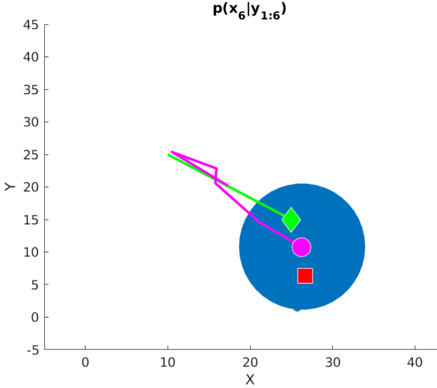

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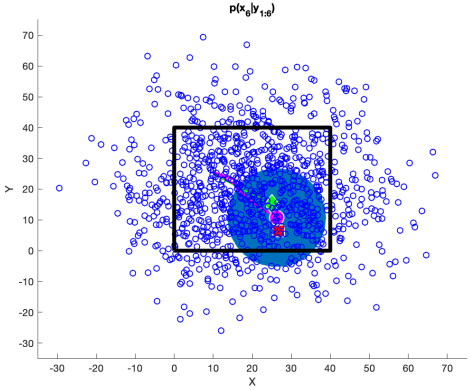
Standard Particle Filter (no resampling)

Only one particle left. What happened to all other particles?



The phenomenon is called depletion!

Particle Filter Depletion

- Depletion means that one or a few particles get almost all probability mass. The particles should approximate the filtering density, which is then not the case.
- The effective number of samples, N_{eff} is a measure of depletion. $N_{\text{eff}} = N$ means that all particles contribute equally, and $N_{\text{eff}} = 1$ means that only one has a non-zero weight.
- Too few design parameters, more degrees of freedom:
  - Sequential importance sampling: means that you only resample when needed, $N_{\text{eff}} < N_{\text{th}}$.
  - The theory allows for a general proposal distribution $q(x_k^{(i)}|x_{0:k-1}^{(i)}, y_{1:k})$ for how to sample a new state in the time update. The "prior" $q(x_k^{(i)}|x_{0:k-1}^{(i)}, y_{1:k}) = p(x_k^{(i)}|x_{k-1}^{(i)})$ is the standard option, but there might be better ones.
- One simple trial and error trick: dithering. Increase process noise and measurement noise. In the introductory example, having a larger R in the filter would avoid depletion (but decrease accuracy)

The Effective Number of Samples N_{eff}

Define N_{eff} as a measure of how many particles are actually contributing to the estimate

$$N_{\text{eff}} \stackrel{\text{def}}{=} \frac{N}{1 + N^2 \operatorname{Var}(\omega^{(i)})} = \begin{cases} N, & \text{if } \operatorname{Var}(\omega^{(i)}) = 0 \\ \approx 0, & \text{if } \operatorname{Var}(\omega^{(i)}) \text{ is large} \end{cases}$$

This can be approximated by

$$\hat{N}_{\text{eff}} = \frac{1}{\sum_{i=1}^{N} (\omega^{(i)})^2} = \begin{cases} N, & \text{if } \omega^{(i)} = \frac{1}{N} \quad \forall i \\ 1, & \text{if } \omega^{(j)} = 1,\ \omega^{(i)} = 0 \quad \forall i \neq j \end{cases}$$

This number can and should be monitored whenever the particle filter is used!

SIR and SIS Particle Filter

- Resampling is necessary and generally good.
- Resamplng also adds variance to the weights, which decreases N_{eff}
- Do we need to resample every time? No!
  1. SIR PF: Always resample in step 3 $\Rightarrow$
  2. SIS PF: Only resample in step 3 if $N_{\text{eff}} < N_{\text{th}}$. A good starting value is $N_{\text{th}} = 2N/3$.

Particle Filter Convergence

How good is the PF approximation

$$p(x_k|y_{1:k}) \approx \sum_{i=1}^{N} \omega_{k|k}^{(i)} \delta(x_k - x_{k|k}^{(i)}) = \hat{p}$$

Main theoretical result

$$\|p - \hat{p}\| < \frac{C g_k}{N}$$

where

- C — constant
- N — number of particles
- g_k — is a polynomial in time that grows quite quickly both with time and with the state dimension

This is not a strong result. In practice, the result might be much better than this conservative bound.

Particle Filter Proposals I

- The main cause of the depletion in the introductory example was the large process noise.
- The new particles after the time update are far away from the measurement, and gets a small likelihood.
- Key idea here: We can reverse the roles of the prior and likelihood!
- General fact for the sampling step 4: it is possible to use a general proposal distribution, $q(x_k|x_{k-1}^{(i)}, y_k)$:

$$x_k^{(i)} \sim q(x_k|x_{k-1}^{(i)}, y_k)$$

$$\omega_{k|k-1}^{(i)} \propto \omega_{k-1|k-1}^{(i)} \frac{p_v\left(x_k^{(i)} - f(x_{k-1}^{(i)})\right)}{q(x_k^{(i)}|x_{k-1}^{(i)}, y_k)}$$

Particle Filter Proposals II

In the standard PF: sample from prior

$$q(x_k|x_{k-1}^{(i)}, y_k) = p(x_k|x_{k-1}^{(i)}) = p_v(x_k^{(i)} - f(x_k^{(i)})) = \text{“prior”}$$

Main alternative: sample from likelihood

$$q(x_k|x_{k-1}^{(i)}, y_k) = p(y_k|x_k) = p_e(y_k - h(x_k)) = \text{“likelihood”} \Rightarrow$$

This alternative includes an often nontrivial implicit sampling

$$x_k^{(i)} \sim p_e(y_k - h(x_k^{(i)})),$$

and the state space model is used to update the weights

$$\omega_{k|k-1}^{(i)} \propto \omega_{k-1|k-1}^{(i)} \frac{p_v(x_k^{(i)} - f(x_{k-1}^{(i)}))}{p_e(y_k - h(x_k^{(i)}))}$$

SIS PF Algorithm

Choose the number of particles N, a proposal density $q(x_k^{(i)}|x_{0:k-1}^{(i)}, y_{1:k})$, and a threshold N_{th} (for instance $N_{th} = \frac{2}{3}N$).

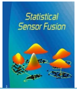
- *Initialization:* Generate $x_0^{(i)} \sim p_{x_0}$ and $\omega_{1|0}^{(i)}$, $i = 1, \dots, N$.

Iterate for $k = 1, 2, \dots$:

1. *Measurement update:* For $i = 1, 2, \dots, N$:
   $w_{k|k}^{(i)} \propto w_{k|k-1}^{(i)} p(y_k|x_k^{(i)})$, normalize $w_{k|k}^{(i)}$.
2. *Estimation:* MMSE $\hat{x}_{k|k} \approx \sum_{i=1}^{N} w_{k|k}^{(i)} x_{k|k}^{(i)}$.
3. *Resampling:* Resample with replacement when $N_{\text{eff}} = \frac{1}{\sum_i (w_{k|k}^{(i)})^2} < N_{th}$.
4. *Prediction:* Generate samples $x_{k+1}^{(i)} \sim q(x_k|x_{k-1}^{(i)}, y_k)$,
   update the weights $w_{k+1|k}^{(i)} \propto w_{k|k}^{(i)} \frac{p(x_k^{(i)}|x_{k-1}^{(i)})}{q(x_k^{(i)}|x_{k-1}^{(i)}, y_k)}$, normalize $w_{k+1|k}^{(i)}$.

Summary: Particle Filtering Properties

1. Choice of N is a complexity *vs.* performance trade-off. Complexity is linear in N, while the error in theory is bounded as g_k/N, where g_k is polynomial in k but independent of n_x.
2. Depletion denotes the case where the PF does not work: one or a few particles get all the probability.
3. $N_{\text{eff}} = \frac{1}{\sum_i (w_k^{(i)})^2}$ controls how often to resample. Resample if $N_{\text{eff}} < N_{th}$. $N_{\text{th}} = N$ gives SIR. Resampling increases variance in the weights, and thus the variance in the estimate, but it is needed to avoid depletion.
4. The proposal density. An appropriate proposal makes the particles explore the most critical regions, without wasting efforts on meaningless state regions.
5. Pretending the process (and measurement) noise is larger than it is (dithering, jittering, roughening) is as for the EKF and UKF often a sound *ad hoc* solution to avoid filter divergence.



Section 9.4–9.6