# SENSOR FUSION

# Discretizing Motion Models

## Sensor Fusion

**Fredrik Gustafsson**
fredrik.gustafsson@liu.se

Gustaf Hendeby
gustaf.hendeby@liu.se

Linköping University

# Purpose

To make physical models derived in continuous time useful in filtering and sensor fusion applications.

- Given a continuous time physical model $\dot{x}(t) = Ax(t) + Bu(t)$
- How to get a discrete time model $x_{k+1} = Fx_k + Gu_k$?
- General methodology for discretizing (sampling) linear and nonlinear continuous time models.
- Examples from sensor fusion practice to illustrate.

# Solving an ODE

What is the solution to the ODE $\dot{x} = Ax + Bu$?
Same methodology as in scalar case in calculus.

1. Multiply with integrating factor $e^{-At}$ on both sides
2. Note that
$$\frac{d}{dt}\left(e^{-At}x(t)\right) = e^{-At}\left(\dot{x}(t) - Ax(t)\right)$$
3. Then, the solution by integrating both sides of
$$\int_0^t \frac{d}{ds}\left(e^{-As}x(s)\right) = \int_0^t e^{-As}u(s)ds$$
4. The solution is
$$x(t) = e^{At}x(0) + \int_0^t e^{-A(s-t)}u(s)ds$$
5. We get $F = e^{AT}$ and, if $u$ is piece-wise constant, $G = \int_0^T e^{A\tau}\, d\tau\, B$.

# Example 1: Newton's II law (revisited)

Linear motion governed by Newtons II law, $F = ma = m\ddot{X}$.
Using $x = (p, v)^T$,

$$\dot{x} = \begin{pmatrix} v \\ a \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{F}{m} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & \frac{1}{m} \end{pmatrix}^T u = Ax + Bu$$


Solving the ODE over one sampling interval $T$ gives

$$F = e^{AT} = I + AT + \frac{1}{2}A^2T^2 + \cdots = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \quad \{A^2 = 0\}$$

$$G = \int_0^T e^{A\tau}\, d\tau \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} = \int_0^T \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} d\tau \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}T^2 \\ T \end{pmatrix} \frac{1}{m}.$$

# Inter-sample behaviour

How to treat the input $u$ (control signal or process noise) in a linear continuous time model when discretizing

$$\dot{x} = Ax + Bu, \qquad x_{k+1} = Fx_k + Gu_k$$
$$y = Cx + e, \qquad y_k = Hx_k + Ju_k + e_k$$

- $F = e^{AT}$ is the unique solution to the ODE.
- But $G$ depends on the assumption or knowledge of the inter-sample behaviour. Most important assumptions:
  - Piece-wise constant input: ZOH, zero order hold.
  - Piece-wise linear input: FOH, first order hold.
  - Band-limited according the the Nyquist criterium: BIL, the bilinear transformation.
- Use `c2d` in Matlab for converting a linear continuous time model to a linear discrete time model.
- It also holds that $H = C$, the measurement relation does not change with discretization, but note the input leakage term $Ju_k$ that may appear.

# Different Sampled Models of Double Integrator

## Models

$$\dot{x} = Ax + Bu \qquad x_{k+1} = Fx_k + Gu_k$$

$$y = Cx + Du \qquad y_k = Hx_k + Ju_k$$

State: $x = \begin{pmatrix} p(t) \\ v(t) \end{pmatrix}$

| | | | | |
|---|---|---|---|---|
| Continuous time | $A = \begin{pmatrix} 0_n & I_n \\ 0_n & 0_n \end{pmatrix}$ | $B = \begin{pmatrix} 0_n \\ I_n \end{pmatrix}$ | $C = (I_n,\ 0_n)$ | $D = 0_n$ |
| ZOH | $F = \begin{pmatrix} I_n & TI_n \\ 0_n & I_n \end{pmatrix}$ | $G = \begin{pmatrix} \frac{T^2}{2} I_n \\ TI_n \end{pmatrix}$ | $H = (I_n,\ 0_n)$ | $J = 0_n$ |
| FOH | $F = \begin{pmatrix} I_n & TI_n \\ 0_n & I_n \end{pmatrix}$ | $G = \begin{pmatrix} T^2 I_n \\ TI_n \end{pmatrix}$ | $H = (I_n,\ 0_n)$ | $J = \frac{T^2}{6} I_n$ |
| BIL | $F = \begin{pmatrix} I_n & TI_n \\ 0_n & I_n \end{pmatrix}$ | $G = \begin{pmatrix} \frac{T^2}{4} I_n \\ \frac{T}{2} I_n \end{pmatrix}$ | $H = (I_n,\ \frac{T}{2} I_n)$ | $J = \frac{T^2}{2} I_n$ |

# Translational Motion with $n$ Integrators

Translational kinematics models in $n$D, where $p(t)$ denotes:

- Position: $X(t)$, $\left(X(t), Y(t)\right)^T$, or $\left(X(t), Y(t), Z(t)\right)^T$
- Rotation: $\psi(t)$ or $\left(\phi(t), \theta(t), \psi(t)\right)^T$

The signal $w(t)$ is process noise for a pure kinematic model and a motion input signal in position, velocity, and acceleration, respectively, for the case of using sensed motion as an input rather than as a measurement.

| **State, x** | **Continuous time, $\dot{\mathbf{x}}$** | **Discrete time, $\mathbf{x}(\mathbf{t}+\mathbf{T})$** |
|---|---|---|
| $p$ | $w$ | $x + Tw$ |
| $\begin{pmatrix} p \\ v \end{pmatrix}$ | $\begin{pmatrix} 0_n & I_n \\ 0_n & 0_n \end{pmatrix} x + \begin{pmatrix} 0_n \\ I_n \end{pmatrix} w$ | $\begin{pmatrix} I_n & TI_n \\ 0_n & I_n \end{pmatrix} x + \begin{pmatrix} \frac{T^2}{2} I_n \\ TI_n \end{pmatrix} w$ |
| $\begin{pmatrix} p \\ v \\ a \end{pmatrix}$ | $\begin{pmatrix} 0_n & I_n & 0_n \\ 0_n & 0_n & I_n \\ 0_n & 0_n & 0_n \end{pmatrix} x + \begin{pmatrix} 0_n \\ 0_n \\ I_n \end{pmatrix} w$ | $\begin{pmatrix} I_n & TI_n & \frac{T^2}{2} I_n \\ 0_n & I_n & TI_n \\ 0_n & 0_n & I_n \end{pmatrix} x + \begin{pmatrix} \frac{T^3}{6} I_n \\ \frac{T^2}{2} I_n \\ TI_n \end{pmatrix} w$ |

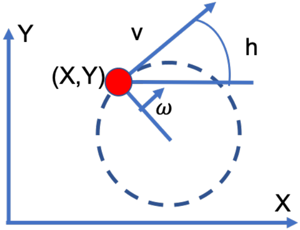
# Nonlinear models

| Classification | Nonlinear | Linear |
|---|---|---|
| Continuous time | $\dot{x} = a(x, u) + v$ <br> $y = c(x, u) + e$ | $\dot{x} = Ax + Bu + v$ <br> $y = Cx + Du + e$ |
| Discrete time | $x_{k+1} = f(x, u) + \bar{v}$ <br> $y = h(x, u) + e$ | $x_{k+1} = Fx + Gu + \bar{v}$ <br> $y = Hx + Ju + e$ |

- Nonlinear filters require a discrete time model.
- The Kalman filter requires a linear discrete time model.
- There are two paths from a nonlinear continuous time model to a linear discrete time model:
  - Discretized linearization: LInearize first, then apply the explicit discretization formulas.
  - Linearized discretization: Try to discretize first, and then linearize.

# Exact Discretization of Coordinated Turn Models

- Coordinated turn models popular in target tracking applications.
- Good compromize between model flexibility and simplicity.
- Possibility of exact sampling one reason for its success.
- Exact sampling even possible for two different choices of state vectors:
  - *Polar velocity* with speed $v$ and heading $h$ as states.
  - *Cartesian velocity* with $v^X, v^Y$ as states.



# Coordinated Turns in 2D World Coordinates

| Cartesian velocity | Polar velocity |
|---|---|
| $\dot{X} = v^X$<br>$\dot{Y} = v^Y$<br>$\dot{v}^X = -\omega v^Y$<br>$\dot{v}^Y = \omega v^X$<br>$\dot{\omega} = 0$ | $\dot{X} = v\cos(h)$<br>$\dot{Y} = v\sin(h)$<br>$\dot{v} = 0$<br>$\dot{h} = \omega$<br>$\dot{\omega} = 0$ |
| $A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\omega & -v^Y \\ 0 & 0 & \omega & 0 & v^X \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ | $A = \begin{pmatrix} 0 & 0 & \cos(h) & -v\sin(h) & 0 \\ 0 & 0 & \sin(h) & v\cos(h) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ |
| $X_{t+T} = X + \frac{v^X}{\omega}\sin(\omega T) - \frac{v^Y}{\omega}(1-\cos(\omega T))$<br>$Y_{t+T} = Y + \frac{v^X}{\omega}(1-\cos(\omega T)) + \frac{v^Y}{\omega}\sin(\omega T)$<br>$v^X_{t+T} = v^X\cos(\omega T) - v^Y\sin(\omega T)$<br>$v^Y_{t+T} = v^X\sin(\omega T) + v^Y\cos(\omega T)$<br>$\omega_{t+T} = \omega$ | $X_{t+T} = X + \frac{2v}{\omega}\sin(\frac{\omega T}{2})\cos(h + \frac{\omega T}{2})$<br>$Y_{t+T} = Y - \frac{2v}{\omega}\sin(\frac{\omega T}{2})\sin(h + \frac{\omega T}{2})$<br>$v_{t+T} = v$<br>$h_{t+T} = h + \omega T$<br>$\omega_{t+T} = \omega$ |

# Summary

| Classification | Nonlinear | Linear |
|---|---|---|
| Continuous-time | $\dot{x} = a(x,u) + v$<br>$y = c(x,u) + e$ | $\dot{x} = Ax + Bu + v$<br>$y = Cx + Du + e$ |
| Discrete-time | $x_{k+1} = f(x,u) + v$<br>$y = h(x,u) + e$ | $x_{k+1} = Fx + Gu + v$<br>$y = Hx + Ju + e$ |

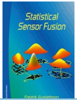
- Discetized linearization: Linearize

$$A = a'_x(x,u), \quad B = a'_u(x,u), \quad C = c'_x(x,u), \quad D = c'_u(x,u),$$

  and sample: $F = e^{AT}$, $G = \int_0^T e^{At} dt B$ (ZOH), $H = C$ and $J = D$.
- Linearized discretization: Sample by solving (if and when possible) the integral

$$x(t+T) = f(x(t), u(t)) = \int_t^{t+T} a(x(\tau), u(\tau)) d\tau,$$

  and then linearize using $F = f'_x(x_k, u_k)$ and $G = f'_u(x_k, u_k)$.




Chapter 12