# Sensor Networks NLS

## Sensor Fusion

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# Localization in Sensor Networks

- Nonlinear estimation applies to a wide range of problems in signal processing, model estimation and sensor fusion.
- Localization in radio networks is an important application
- Only a few nonlinear models appear in practice
- They all enable very concrete formulas for gradients, algorithms and performance bounds

# Sensor Models in Sensor Networks

## Radio Propagation Model

Received signal $y_k(t)$ is a noisy, delayed and attenuated version of the transmitted signal $s(t)$

$$y_k(t) = a_k s(t - \tau_k) + e_k(t), \quad k = 1, 2, \dots, N.$$

How the delay $\tau$ is estimated using known training signal (pilot symbols) is explained in other courses (Signal Processing, Signal Theory, Communication Theory).
What is important is that the delay can be related to the range between the transmitter and receiver

$$r_k = \tau_k v = \|x - p_k\|_2 = \sqrt{(x_1 - p_{k,1})^2 + (x_2 - p_{k,2})^2},$$

where $v$ is the speed of the medium (light for radio, sound for acoustic and water for sonar signals).

# Sensor Models in Sensor Networks

Different use cases:

- *Time-of-arrival* (TOA) — transport delay, transmission time is known to receiver. Then the arrival time is proportional to range $\tau_k = \frac{1}{v}\|x - p_k\|_2$
- *Time-difference-of-arrival* (TDOA) — arrival time known at each receiver (synchronized), but the transmission time not. Then the arrival time is proportional to range plus a bias $\tau_k = \frac{1}{v}\|x - p_k\|_2 + \frac{r_0}{v}$
- Multiple closely separated receivers can convert their TDOA measurements to an angle. This is called Direction-of-arrival (DOA) $\varphi_k$. For two receivers, $\varphi_k = \arccos(\tau_2 - \tau_1)$.
- Estimation of $a_k$ in the signal model gives *received signal strength* (RSS), which does not require any timing or known training signal, just transmitter power $P_0$ and path propagation constant $\alpha$. Then,

$$P_k = P_0 - \alpha \log(\|x - p_k\|).$$

# Basic Network Sensor Models

The basic network measurements in any network (radio, acoustic, sonar, seismic) can be summarized as follows:

**Sensor Network Models**

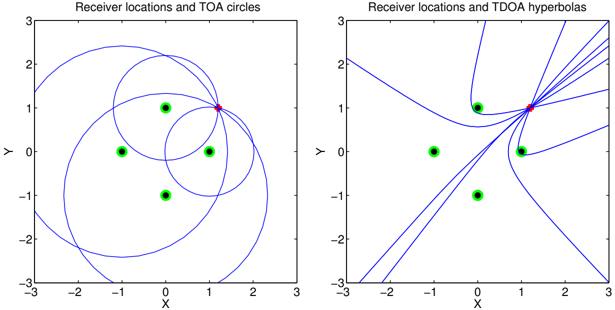
$$\begin{aligned}
\text{TOA} \quad & h_k(x) = r_k = \|x - p_k\| \\
\text{TDOA} \quad & h_k(x) = r_k = \|x - p_k\| + r_0 \\
\text{DOA} \quad & h_k(x) = \varphi_k = \operatorname{arctan2}\left(x_2 - p_{k,2}, x_1 - p_{k,1}\right) \\
\text{RSS} \quad & h_k(x) = P^0 - \alpha \log\left(\|x - p_k\|\right)
\end{aligned}$$

**Note:**

All models are on the form $y_k = h_k(x) + e_k$.

# THE Example (1/5)

TOA corresponds to circles that intersect at the transmitter, but what is TDOA? It will be shown in another lecture that each pair of TDOA measurements can be interpreted as a hyperbolic function, and all possible pairs give a set of hyperbolas that intersect at the transmitter position.




**Note:**

TDOA hyperbolas intersect with poor geometry far away from the network!

# Estimation Criteria

General problem formulation and solution:

$$\hat{x} = \arg\min_x V(x).$$

Summary from Chapter 3:

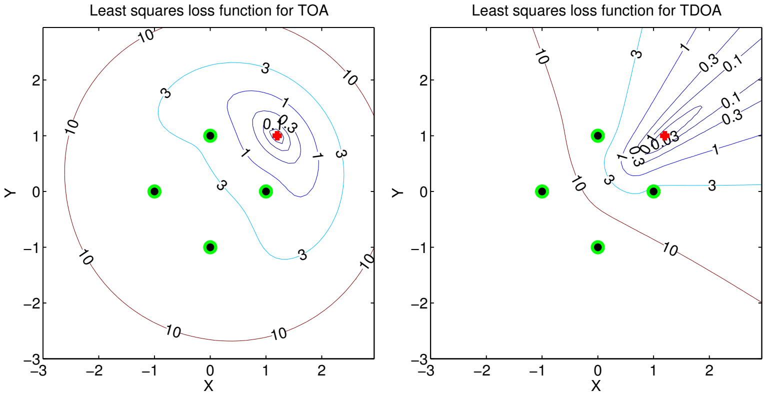
## Loss functions

$$\begin{aligned}
\textbf{NLS}: \quad & V^{NLS}(x) = \|\mathbf{y} - \mathbf{h}(x)\|^2 = \big(\mathbf{y} - \mathbf{h}(x)\big)^T \big(\mathbf{y} - \mathbf{h}(x)\big) \\
\textbf{NWLS}: \quad & V^{NWLS}(x) = \big(\mathbf{y} - \mathbf{h}(x)\big)^T \mathbf{R}^{-1}(x) \big(\mathbf{y} - \mathbf{h}(x)\big) \\
\textbf{ML}: \quad & V^{ML}(x) = \log p_{\mathbf{e}}\big(\mathbf{y} - \mathbf{h}(x)\big) \\
\textbf{GML}: \quad & V^{GML}(x) = \big(\mathbf{y} - \mathbf{h}(x)\big)^T \mathbf{R}^{-1}(x) \big(\mathbf{y} - \mathbf{h}(x)\big) + \log\det \mathbf{R}(x)
\end{aligned}$$

We have here allowed the covariance $R(x)$ to depend on the parameters in the NWLS and Gaussian ML.

# THE Example (2/5)

Level curves $V^{NLS}(x)$ for TOA and TDOA.




Level curves confirm that TDOA has a poor geometry outside the network. TOA, on the other hand, gives an elliptically shaped loss function close to the true position.

# Estimation Methods

## General principles:

***Steepest descent*** *(Stochastic gradient)*

$$\hat{x}_k = \hat{x}_{k-1} + \mu_k \mathbf{H}^T(\hat{x}_{k-1}) R^{-1}(y - \mathbf{h}(\hat{x}_{k-1}))$$

***Gauss-Newton***

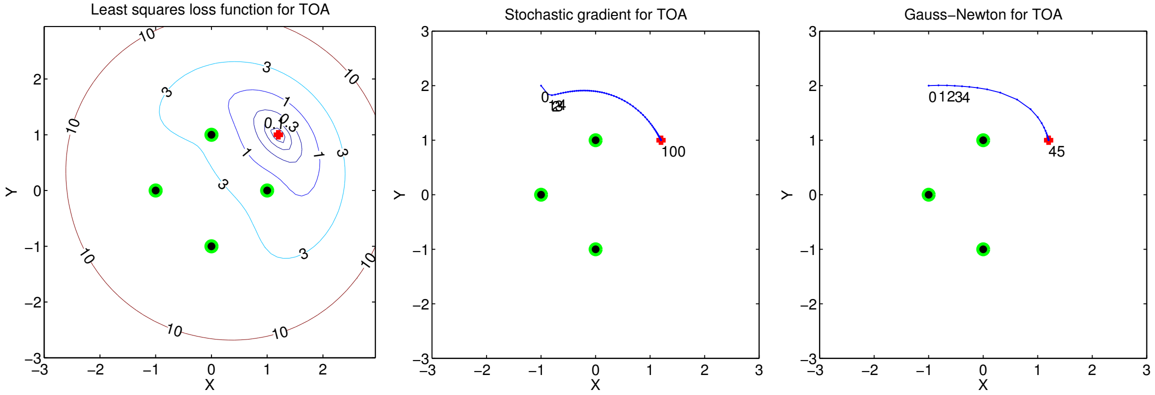
$$\hat{x}_k = \hat{x}_{k-1} + \mu_k \left(\mathbf{H}^T(\hat{x}_{k-1}) R^{-1} \mathbf{H}(\hat{x}_{k-1})\right)^{-1} \mathbf{H}^T(\hat{x}_{k-1}) R^{-1} \left(y - \mathbf{h}(\hat{x}_{k-1})\right)$$

## Problem specific quantities:

| Method | $h(x, p_i)$ | $\partial h / \partial x_1$ | $\partial h / \partial x_2$ |
|---|---|---|---|
| **RSS** | $P_0 + 10\beta \log_{10} r_i$ | $\frac{10\beta}{\log 10} \frac{x_1 - p_{i,1}}{r_i^2}$ | $\frac{10\beta}{\log 10} \frac{x_2 - p_{i,2}}{r_i^2}$ |
| **TOA** | $r_i$ | $\frac{x_1 - p_{i,1}}{r_i}$ | $\frac{x_2 - p_{i,2}}{r_i}$ |
| **TDOA** | $r_i - r_j$ | $\frac{x_1 - p_{i,1}}{D_i} - \frac{x_1 - p_{j,1}}{D_j}$ | $\frac{x_2 - p_{i,2}}{D_i} - \frac{x_2 - p_{j,2}}{D_j}$ |
| **AOA** | $\alpha_i + \arctan \frac{x_2 - p_{i,2}}{x_1 - p_{i,1}}$ | $\frac{-(x_1 - p_{i,1})}{r_i^2}$ | $\frac{x_2 - p_{i,2}}{r_i^2}$ |

# THE Example (3/5)

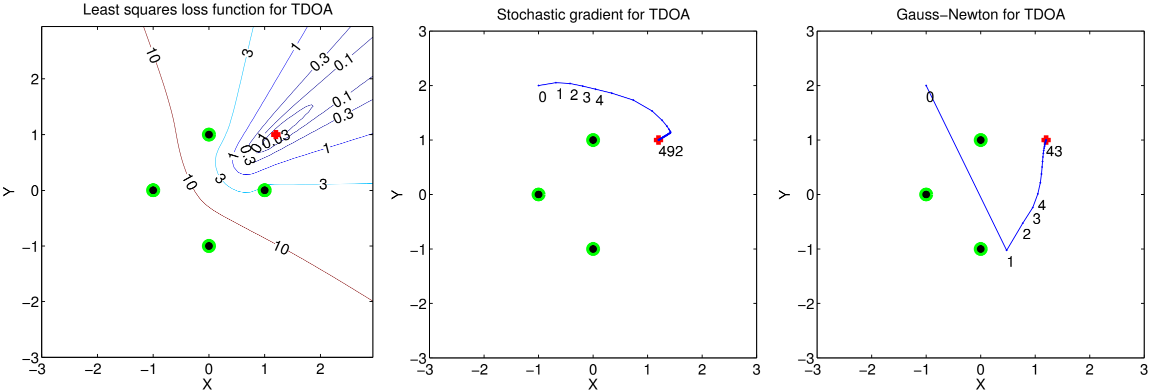
The steepest descent and Gauss-Newton algorithms for TOA.




Both algorithms approache the optimum nicely. The step length is deliberately chosen very small, so it takes 100 and 45, respectively, iterations to reach the optimum.

# THE Example (4/5)

The steepest descent and Gauss-Newton algorithms for TDOA.




The stochastic gradient algorithm is particularly ineffective in the end, where it needs hundreds of iteration to climb a shallow ridge.

# CRLB Estimation Bound

The Cramér-Rao Lower Bound (CRLB) provides a lower bound on the covariance of any unbiased estimator

## CRLB Estimation Bound

$$\mathrm{Cov}(\hat{x}) \geq \mathcal{I}^{-1}(x^0),$$

where $\mathrm{E}(\hat{x}) = x^0$ (unbiased estimator) and $\mathcal{I}(x)$ is the Fisher Information Matrix (FIM)

The FIM is defined from the likelihood as

$$\mathcal{I}_k(x) = \mathrm{E}\left[\left(\frac{d \log p(y_k|x)}{dx}\right)\left(\frac{d \log p(y_k|x)}{dx}\right)^T\right].$$

which is additive so $\mathcal{I}(x) = \sum_k \mathcal{I}_k(x)$.

# CRLB Localization Bound

If $x$ is the position (in 1D, 2D, 3D or generally $n$D), then the Mean Square Error (MSE) can be written

$\mathrm{E}\left((x-\hat{x})^T(x-\hat{x})\right) = \mathrm{tr}\,\mathrm{E}\left((x-\hat{x})(x-\hat{x})^T\right) = \mathrm{tr}(\mathrm{Cov}(\hat{x})) = P_{11} + P_{22} + \cdots + P_{dd}.$

Then we get the following MSE bound

## CRLB Localization Bound

$$\mathrm{MSE} \geq \mathrm{tr}\left(\mathcal{I}^{-1}(x^0)\right),$$

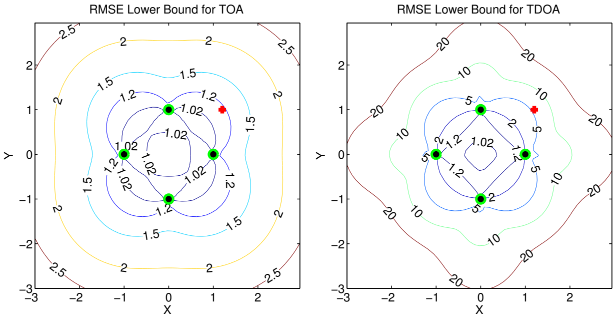
where $\mathrm{E}\left(\hat{x}\right) = x^0$ (unbiased estimator) and $\mathcal{I}(x)$ is the Fisher Information Matrix (FIM)

## Note:

This can be used to get a lower bound on the position standard deviation (RMSE) from any network configuration.

# THE Example (5/5)

## CRLB for TOA and TDOA:




**Note:**

- Inside the convex hull of the network, the lower bound is very similar
- Outside the convex hull, TDOA performance degrades quickly!

# Summary

- Signal model for localization in sensor networks

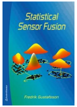
$$\mathbf{y} = \mathbf{h}(x - \mathbf{p}) + \mathbf{e}, \quad \mathrm{Cov}(\mathbf{e}) = \mathbf{R}$$

  $x$ is the unknown position, $\mathbf{p}$ is the (known in this chapter) sensor locations.
- The basic network measurements:

$$\begin{aligned} \text{TOA} \quad r_k &= \|x - p_k\| + e_k \\ \text{TDOA} \quad r_k &= \|x - p_k\| + r_0 + e_k \\ \text{DOA} \quad \varphi_k &= \text{arctan2}(x_2 - p_{k,2}, x_1 - p_{k,1}) + e_k \\ \text{RSS} \quad y_k &= P_0 - \beta \log(\|x - p_k\|) \end{aligned}$$

- NLS approaches for estimating $x$.
- CRLB theorem implies a lower bound on any unbiased estimator of position $\hat{x}$,

$$\mathrm{MSE} = \text{tr}\left(\mathrm{Cov}(\hat{x})\right) \succeq \text{tr}\left(\mathcal{I}^{-1}(x^0)\right),$$




Sections 4–4.2