SENSOR FUSION

Weighted Least Squares

Sensor Fusion

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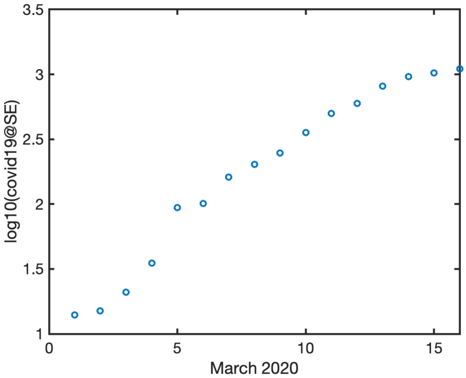
Linear Regression

Consider the time series in the plot below



Fitting a straight line to the data points can be formulated as the linear regression problem

$$y_k = x_1 + kx_2 + e_k = \begin{pmatrix}1, & k\end{pmatrix} x + e_k = H_k x + e_k$$

Least Squares Estimate

Multiply $y_k = H_k x + e_k$ with H_k^T from the left and sum over k

$$\sum_{k=1}^{N} H_k^T y_k = \sum_{k=1}^{N} H_k^T H_k x + \sum_{k=1}^{N} H_k^T e_k$$

We can solve for x to get an *estimate*

$$\hat{x} = \left(\sum_{k=1}^{N} H_k^T H_k \right)^{-1} \sum_{k=1}^{N} H_k^T y_k$$

The solution can also be derived by minimizing the LS cost function

$$V^{LS}(x) = \sum_{k=1}^{N} \| y_k - H_k x) \|^2$$

Weighted Least Squares Estimate

Weighted least squares: multiply from the left with $H_k^T R_k^{-1}$ instead, which gives

$$\hat{x} = \left(\sum_{k=1}^{N} H_k^T R_k^{-1} H_k \right)^{-1} \sum_{k=1}^{N} H_k^T R_k^{-1} y_k$$

This corresponds to minimizing the cost function

$$V^{WLS}(x) = \sum_{k=1}^{N} (y_k - H_k x)^T R_k^{-1} (y_k - H_k x)$$

WLS Covariance

If we assume there is a 'true' parameter x^o in the model $y_k = H_k x^o + e_k$ (no model error), then the estimation error is

$$\hat{x} - x^o = \left(\sum_{k=1}^{N} H_k^T R_k^{-1} H_k \right)^{-1} \sum_{k=1}^{N} H_k^T R_k^{-1} e_k$$

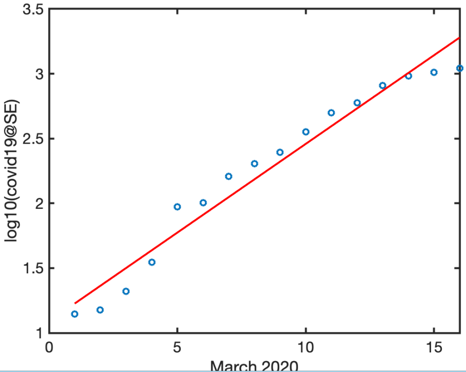
The covariance P is by definition and algebraic simplifications (note that $\mathrm{E}\left(e_k e_l^T\right) = 0$ if $k \neq l$)

$$\begin{aligned} P &= \mathrm{Cov}(\hat{x}) = \mathrm{E}\left(\hat{x} - x^o\right)\left(\hat{x} - x^o\right)^T \\ &= \left(\sum_{k=1}^{N} H_k^T R_k^{-1} H_k \right)^{-1} \sum_{k=1}^{N} H_k^T R_k^{-1} \mathrm{E}\left(e_k e_k^T\right) R_k^{-1} H_k \left(\sum_{k=1}^{N} H_k^T R_k^{-1} H_k \right)^{-1} \\ &= \left(\sum_{k=1}^{N} H_k^T R_k^{-1} H_k \right)^{-1} \end{aligned}$$

Least Squares Example

Matlab code for computing the LS estimate of the parameter and corresponding straight line

```
y=log10([14 15 21 35 94 101 161 203 248 355 500 599 814 961 1022 1103]');
H=[ones(16,1) (1:16)'];
xhat=inv(H'*H)*(H'*y);
yhat=H*xhat;
plot(1:16,y,'o-',1:16,yhat,'-')
```



WLS in batch form

Eliminate the index (time or space usually) k, which leads to compact expressions

Linear model:

$$\begin{aligned} y_k &= H_k x + e_k, \quad \mathrm{Cov}(e_k) = R_k, \quad k = 1, \dots, N, \\ \mathbf{y} &= \mathbf{H}x + \mathbf{e}, \quad \mathrm{Cov}(\mathbf{e}) = \mathbf{R}. \end{aligned}$$

WLS loss function

$$V^{WLS}(x) = \sum_{k=1}^{N} (y_k - H_k x)^T R_k^{-1} (y_k - H_k x) = (\mathbf{y} - \mathbf{H}x)^T \mathbf{R}^{-1} (\mathbf{y} - \mathbf{H}x).$$

Solution in batch form

$$\begin{aligned} \hat{x} &= \left(\sum_{k=1}^{N} H_k^T R_k^{-1} H_k \right)^{-1} \sum_{k=1}^{N} H_k^T R_k^{-1} y_k = (\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H})^{-1} \mathbf{H}^T \mathbf{R}^{-1} \mathbf{y}, \\ P &= \left(\sum_{k=1}^{N} H_k^T R_k^{-1} H_k \right)^{-1} = (\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H})^{-1}. \end{aligned}$$

Sequential WLS

The WLS estimate can be computed recursively in the space/time sequence y_k. Suppose the estimate $\hat{x}_{k-1}$ with covariance P_k are based on observations $y_{1:k-1}$, where we initiate with $\hat{x}_0$ and P_0 (a 'prior'). A new observation is fused using

$$\hat{x}_k = \hat{x}_{k-1} + P_{k-1}H_k^T\left(H_kP_{k-1}H_k^T + R_k\right)^{-1}(y_k - H_k\hat{x}_{k-1}),$$

$$P_k = P_{k-1} - P_{k-1}H_k^T\left(H_kP_{k-1}H_k^T + R_k\right)^{-1}H_kP_{k-1}.$$

This is useful for recursive estimation when sensor information comes at different times. It eliminates the need to save old sensor data.

Summary WLS

Linear model on sequential and batch forms, respectively:

$$y_k = H_k x + e_k, \quad \mathbf{y} = \mathbf{H} x + \mathbf{e}$$

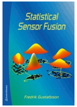
WLS loss function

$$V^{WLS}(x) = \sum_{k=1}^{N} (y_k - H_k x)^T R_k^{-1} (y_k - H_k x) = (\mathbf{y} - \mathbf{H} x)^T \mathbf{R}^{-1} (\mathbf{y} - \mathbf{H} x).$$

WLS solution

$$\hat{x} = \left(\sum_{k=1}^{N} H_k^T R_k^{-1} H_k \right)^{-1} \sum_{k=1}^{N} H_k^T R_k^{-1} y_k = (\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H})^{-1} \mathbf{H}^T \mathbf{R}^{-1} \mathbf{y},$$

$$P = \left(\sum_{k=1}^{N} H_k^T R_k^{-1} H_k \right)^{-1} = (\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H})^{-1}.$$



Sections 2–2.2