Solution for TSRT09 Control Theory, 2024-03-22

1. (a) See Section 9.4 of the book.

   (b) Compute the Jacobian linearization at $x = 0$:

   $$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix}$$

   Solving the characteristic equation

   $$\det(\lambda I - A) = \lambda^2 + 2\lambda - 1 = 0$$

   gives $\lambda_{1,2} = -1 \pm \sqrt{2}$. Since $\mathrm{Re}(\lambda_1) < 0$ and $\mathrm{Re}(\lambda_2) > 0$, the equilibium $x = 0$ is a saddle point.

   (c) Differentiate the output

   $$\begin{aligned} y &= x_1 \\ \dot{y} &= x_1^2 + x_2 \\ \ddot{y} &= 2x_1^3 + 2x_1x_2 + u \end{aligned}$$

   and then choose as feedback law

   $$u = -2x_1^3 - 2x_1x_2 - x_1^2 - x_2 - 3x_1 + r$$

   (d) For small ϵ we have

   $$|S(i\omega)| < \epsilon \iff |G(i\omega)F_y(i\omega)| > 1/\epsilon; \qquad |T(i\omega)| < \epsilon \iff |G(i\omega)F_y(i\omega)| < \epsilon$$

   meaning that it should be

   - $|G(i\omega)F_y(i\omega)| > 100$ for $\omega \leq 1$ rad/s
   - $|G(i\omega)F_y(i\omega)| < 0.01$ for $\omega > 120$ rad/s

2. (a) The pole polynomial is $p(s) = (s-1)(s+1)(s+3)$, hence the poles are $s = -3, -1, +1$, all with multiplicity 1. There are no zeros.

   (b) The singular values are the square roots of the eigenvalues of $G^*(i\omega)G(i\omega)$. As G is 2×1, G^*G is 1×1, i.e., there is only one singular value, shown in Fig. 1, left.

   (c) The first component of $G(s)$ is unstable, so the main priority is to stabilize that.

   (d) This can be achieve trivially with a P-regulator $F_y = [k_1 \ k_2]$ and $F_r = I_2$. For instance choosing $k_1 = 1$ and $k_2 = 0.1$, one gets that the closed loop transfer function `Gc=minreal(feedback(G*Fy,eye(2)))` is

   $$G_c = \frac{1}{(s+3.24)(s^2+0.86s+1.914)} \begin{bmatrix} (s+3)^2 & 0.1(s+3)^2 \\ (s-2)(s-1) & 0.1(s-2)(s-1) \end{bmatrix}$$

   To check stability it is enough to notice that the poles in closed loop are $s = -0.43 \pm 1.3148i, -3.24$. The step response is shown in Fig. 1, right.

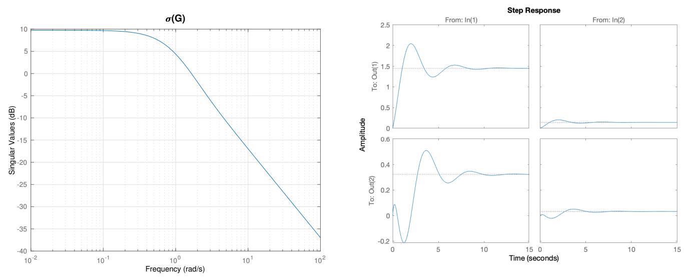

Figure 1: Ex. 2. Left: Singular value; right: step responses.

3. (a) Computing the state covariance Π_x means solving the Lyapuov equation $A\Pi_x + \Pi_x A^T + \rho_1 N N^T = 0$, where $\rho_1 =$ covariance of v_1. Denoting $\Pi_x = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$ and inserting the values of A and N into the Lyapunov equation, one gets

$$2p_1 = \rho_1, \qquad 3p_2 = \rho_1, \qquad 4p_3 = \rho_1 \qquad \Longrightarrow \qquad \Pi_x = \rho_1 \begin{bmatrix} 1/2 & 1/3 \\ 1/3 & 1/4 \end{bmatrix}$$

(b) In the Laplace domain

$$Y(s) = \begin{bmatrix} G_1(s) & 1 \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix}$$

where the transfer function from v_1 to y is

$$G_1(s) = C(sI - A)^{-1}B = \frac{2s+3}{(s+1)(s+2)}$$

Since v_1 and v_2 are uncorrelated, their cross-spectrum is 0, and the spectrum of the output is

$$\begin{aligned} \Phi_y(\omega) =& \begin{bmatrix} G_1(i\omega) & 1 \end{bmatrix} \begin{bmatrix} \Phi_{v_1}(\omega) & 0 \\ 0 & \Phi_{v_2}(\omega) \end{bmatrix} \begin{bmatrix} G_1^*(i\omega) \\ 1 \end{bmatrix} \\ =& \rho_1 |G_1(i\omega)|^2 + \rho_2 = \rho_1 \frac{(9+4\omega^2)}{(1+\omega^2)(4+\omega^2)} + \rho_2 \end{aligned}$$

(c) The Kalman filter for the system is

$$\dot{\hat{x}} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \hat{x} + K \left(y - \begin{bmatrix} 1 & 1 \end{bmatrix} \hat{x} \right)$$

where $K = \frac{1}{\rho_2} P \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with $P = P^T > 0$ solving the associated ARE. With $\rho_1 = \rho_2 = 1$ it is

$$P = \begin{bmatrix} 0.3322 & 0.2470 \\ 0.2470 & 0.2000 \end{bmatrix} \quad \text{and hence} \quad K = \begin{bmatrix} 0.5792 \\ 0.4471 \end{bmatrix}$$

4. (a) The extended system is

$$G_e = \begin{bmatrix} 0 & W_u \\ 0 & W_T G \\ W_S & W_S G \\ 1 & G \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{s+1} \\ 0 & \frac{(s+1)(s-3)}{(s+0.1)(s+2)(s+4)} \\ \frac{2}{s+1} & \frac{2(s-3)}{(s+0.1)(s+1)(s+2)} \\ 1 & \frac{s-3}{(s+0.1)(s+2)} \end{bmatrix}$$

(b) The $\mathcal{H}_\infty$ problem is feasible for $\gamma = 1$ and $\gamma = 5$, but not for $\gamma = 0.5$.

(c) We obviously choose $\gamma = 1$ because it corresponds to a lower impact of the disturbance on the system.

(d) The regulator associated to $\gamma = 1$ is

$$F_y = \frac{-198.64(s+4)(s+2)(s+0.1)}{(s+3.916)(s+1.045)(s^2+36.19s+570.3)}$$

S, T and G_{wu} for the $\mathcal{H}_\infty$ design are shown in Fig. 2.

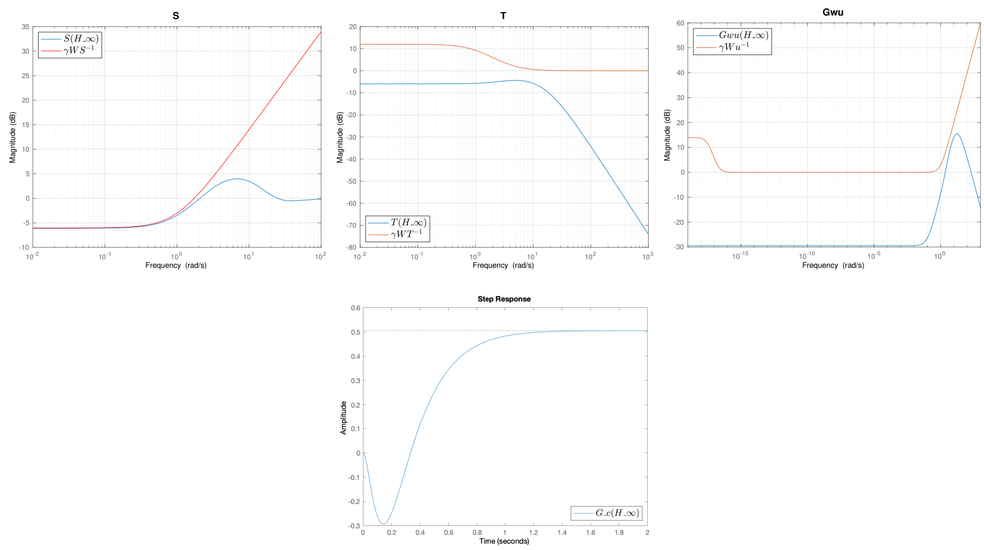

Figure 2: $\mathcal{H}_\infty$ loopshaping design for $\gamma = 1$ in Ex. 4. Lower plot: step response.

(e) The plant G is non-minimum phase: it has a zero in the right half plane. Such zero is also in the closed loop system, hence we expect that the step response starts "in the wrong direction", see Fig. 2.

5. (a) The describing function for a saturation is given in the Appendix of Ch 14.

$$Y_f(C) = \begin{cases} \frac{2}{\pi}(\arcsin(\frac{1}{C})) + \sqrt{\frac{1}{C^2} - \frac{1}{C^4}} & C > 1 \\ 1 & C \le 1 \end{cases}$$

Since $Y_f(C)$ is real, we need to find the value of ω at which

$$G(i\omega) = K\frac{-4\omega + i(\omega^2 - 3)}{\omega(\omega^2 + 1)(\omega^2 + 9)}$$

becomes real, which is $\omega = \sqrt{3}$. When C grows, the curve $-\frac{1}{Y_f(C)}$ "starts" at -1 and continues towards the left, see Fig. 3. At $\omega = \sqrt{3}$, it is $|G(i\sqrt{3})| = \frac{K}{12}$, meaning that intersection is possible only if $K \geq 12$, see Fig. 3.

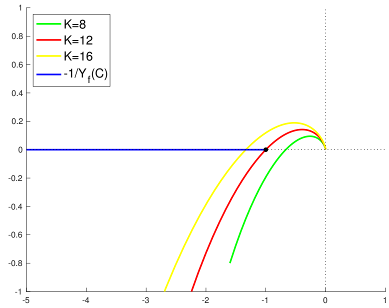

Figure 3: Describing function exercise.

(b) For $K \geq 12$ the intersection of $-\frac{1}{Y_f(C)}$ with $G(i\omega)$ (yellow curve in Fig. 3) is as in Fig. 14.9(a) of the book, namely the self-sustained oscillations are predicted to be stable.

(c) For $K < 12$ the entire curve $-\frac{1}{Y_f(C)}$ is "above" $G(i\omega)$ (green curve in Fig. 3). Hence in this case the situation is similar to that of Fig. 14.10(a) of the book, and leads to vanishing oscillations.