Solution for TSRT09 Control Theory, 2023-08-22

1. (a) A system is non-minimum phase e.g. when it has a zero in the RHP, or a delay (in general, in the SISO case: when the inverse of its transfer function is unstable).

   (b) For perfect disturbance cancelation ($y = 0$) you need $u = -2(s+1)d/(s+3)$. The static gain $d \to u$ in this expression is $2/3$. Since the maximal amplitude of the disturbance is $|d| = 3$, you need an input amplitude of at least 2 for u, which is at odds with the constraint $|u| \leq 1$. Hence the answer is no.

   (c) The pole polynomial is $p(s) = (s+1)(s+3)(s+5)$ and the poles are -1, -3, -5, all with multiplicity 1.

   (d) Use the spectral factorization theorem: given a spectrum $\Phi_v(\omega)$ write it as $\Phi_v(\omega) = |G(i\omega)|^2$, where $G(s)$ is a linear, stable transfer function. Simulate this linear system driven by white noise.

   (e) The Bode integral theorem says that, under the assumption that the loop gain decays at least as $|s|^{-2}$, then it has to have at least an unstable pole.

2. (a)

$$G(0) = \begin{bmatrix} 1/2 & 5 \\ 4/3 & 3/4 \end{bmatrix} \Rightarrow RGA(0) = \begin{bmatrix} -0.0596 & 1.0596 \\ 1.0596 & -0.0596 \end{bmatrix}$$

   The $u_1 \leftrightarrow y_1$, $u_2 \leftrightarrow y_2$ pairing is unsuitable as it corresponds to negative diagonal elements. Instead the other pairing $u_1 \leftrightarrow y_2$, $u_2 \leftrightarrow y_1$ corresponds to off-diagonal elements that are close to 1, hence it should work.

   (b) Using

$$F_{y_2} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

   The pole polynomial for the closed loop system $G_{c_2} = (I + GF_{y_2})^{-1}G$ (taking $F_{r_2} = I$)

$$p(s) = s^4 + 28s^3 + 249s^2 + 854s + 932$$

   gives the poles

$$-14.7607, \ -6.4512, \ -4.7101, \ -2.0780$$

   Using instead F_{y_1} would lead to the closed-loop poles

$$-15.8194, \ -4.0000, \ -2.0978, \ 3.9173$$

   i.e., to an unstable closed-loop system, as expected from RGA.

   (c) Computing $S = (I + GF_{y_2})^{-1}$, one gets the singular values shown in Fig. 1. Constant disturbances and low-frequency disturbance, up to around 1 rad/s, get an attenuation of around 10 dB. The 3 dB threshold is passed at around 5 rad/s.

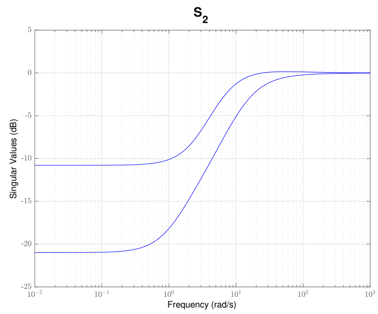


Figure 1: Ex. 2. Singular values of S.

3. (a) The extended system is

$$G_e = \begin{bmatrix} 0 & W_u \\ 0 & W_T G \\ W_S & W_S G \\ 1 & G \end{bmatrix} = \begin{bmatrix} 0 & \frac{10}{s+1} \\ 0 & \frac{(s+2)}{(s+0.2)(s+5)} \\ \frac{K(s+3)}{(s+1)(s+5)} & \frac{K(s+2)(s+3)}{(s+0.2)(s+1)^2(s+5)} \\ 1 & \frac{s+2}{(s+0.2)(s+1)} \end{bmatrix}$$

(b) Since what matters is W_S^{-1}, $K = 10$ pushes down S at low frequencies more than $K = 1$, hence we choose $K = 10$.

(c) The resulting $\mathcal{H}_2$ controller is

$$F_y = \frac{4.84 \cdot 10^6 (s + 4.019)(s + 0.2)}{(s + 1.55 \cdot 10^7)(s + 4.981)(s + 0.7114)}$$

S, T and G_{wu} for the two designs are shown in Fig. 2.

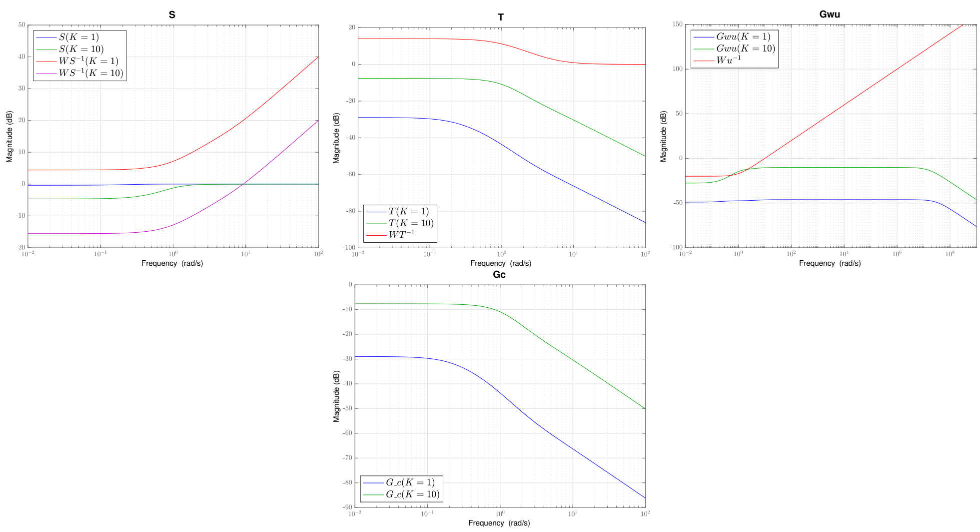
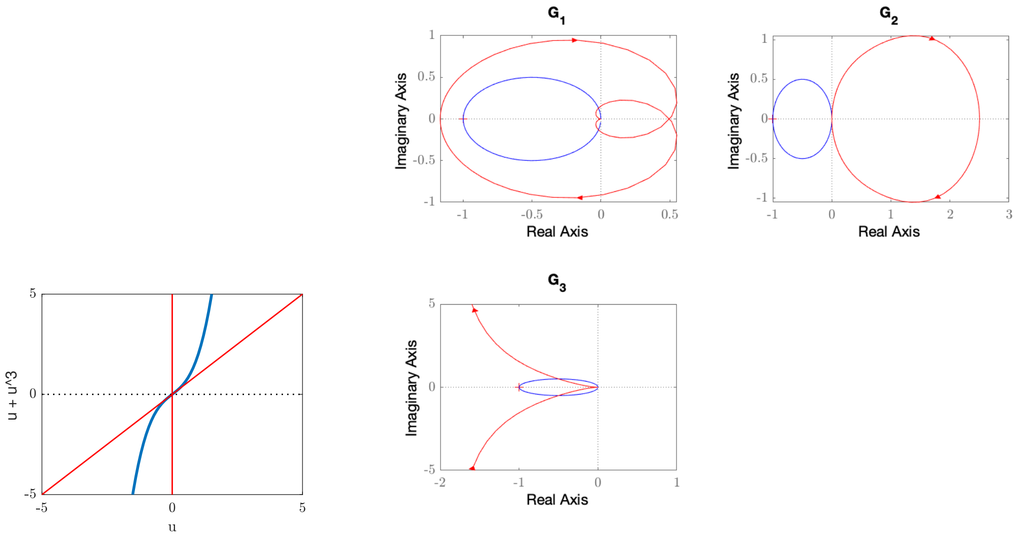


Figure 2: $\mathcal{H}_2$ loopshaping for $K = 1$ (blue) and $K = 10$ (green) design of Ex. 3.

(d) As can be seen in Fig. 2, the $K = 10$ loopshaping (green) corresponds to lower sensitivity at low frequencies, compensated however by a higher complementary sensitivity T and higher G_{wu} at all frequencies. Also the static gain and cross over frequency are higher for the $K = 10$ design, see G_c plot in Fig. 2.

4. The cubic nonlinearity is such that $k_1 = 1 \leq \frac{f(u)}{u} \leq \infty = k_2$, see Fig. 3(a). This means that the "forbidden region" in the circle criterion is a disk passing through -1 and 0 (blue disk in Fig. 3(b)). From Fig. 3(b), only the Nyquist curve of G_2 does not intersects this disk, hence this is the only stable closed loop system. It is so for all values of $K > 0$.



(a) (b)

Figure 3: Ex. 4. (a): The cubic nonlinearity f (blue). The k_1 and k_2 lines are in red. (b): Nyquist diagrams of $G_i(s)$ (red) and disk for f (blue).

5. (a) It is straightforward to check that $x_{\text{eq}} = 0$ is an equilibrium point. The Jacobian linearization of the system at x_{eq} is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

which has eigenvalues $\lambda = 0$ (of multiplicity 2) and $\lambda = 1$. Hence the open loop system is unstable.

(b) The controllability matrix for the linearization

$$S = [B\ AB\ A^2B] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has full rank, hence we can use the Jacobian linearization for local stabilization. One possible gain is $L = \begin{bmatrix} 18 & 7 & 24 \end{bmatrix}$ which places the closed-loop poles of $A - BL$ in $\{-1, -2, -3\}$.

(c) Differentiating the output, for $h_1(x)$ it is

$$\begin{aligned} \dot{y} &= x_1^3 + x_2 \\ \ddot{y} &= 3x_1^5 + 3x_1^2 x_2 + u \end{aligned}$$

meaning that the relative degree is 2, while for $h_2(x)$ it is

$$\begin{aligned} \dot{y} &= x_3 + \sin x_1 \\ \ddot{y} &= x_3 + \sin x_1 + x_1^3 \cos x_1 + x_2 \cos x_1 \\ \dddot{y} &= x_3 + \big(1 - x_1^3(x_1^3 + x_2) - x_2\big) \sin x_1 + \cos x_1 (x_1^3 + x_2)(1 + 3x_1^2) + u \cos x_1 \end{aligned}$$

i.e., the relative degree is 3. Only the latter case leads to an easy feedback linearization, as there is no zero dynamics.