Solution for TSRT09 Control Theory, 2023-06-07

1. (a) See book.

   (b) The system $\dot{x} = -2x + v$ is stable, and therefore we can use Theorem 5.3 in the book, and compute the variance of x, Π_x, as solution of the Lyapunov equation $A\Pi_x + \Pi_x A^T + BRB^T = 0$, where $A = -2$, $B = 1$, and $R = r$. The solution is $\Pi_x = \frac{r}{4}$.

   (c) The observability matrix: $\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix}$ has full rank if and only if $b \neq 0$ (a can be anything).

   (d) The controllability matrix $\mathcal{S} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ has rank 1, hence the system is not controllable. The controllable subspace is given by $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ but the unstable pole does not belong to it. Hence the system is not stabilizable.

   (e) Differentiating: $\dot{y} = \dot{x}_1 + \dot{x}_2 = \sin x_2 + bu + u$. When $b \neq -1$, the input appears. When $b = -1$, it is $\dot{y} = \sin x_2$ and $\ddot{y} = u \cos x_2$, and we have relative degree 2.

2. (a) The minors are $\frac{2}{s+2}$, $\frac{1}{s+1}$, $\frac{4}{s+2}$, $\frac{3}{s+2}$, and $\frac{2(s-1)}{(s+2)^2(s+1)}$. The pole polynomial is $p(s) = (s+2)^2(s+1)$, meaning that the system has two poles in -2.

   (b) $RGA(G(0)) = G(0). * (G^\dagger(0))^T = \begin{pmatrix} -3 & 4 \\ 4 & -3 \end{pmatrix}$. The best pairing is $u_1 \leftrightarrow y_2$ and $u_2 \leftrightarrow y_1$.

   (c) For $U_2 = -KY_2$ one gets $Y_2 = G_{21}U_1 - KG_{22}Y_2 \iff Y_2 = \frac{G_{21}}{1+KG_{22}}U_1$. Inserting this into $Y_1 = G_{11}U_1 - G_{12}KY_2$, one gets $Y_1 = \frac{2s^2+(6+2K)s+4-2K}{(s+2)(s+1)(s+2+3K)}U_1 = \tilde{G}U_1$ and $\tilde{G}(0) = \frac{2-K}{2+3K}$.

   (d) Recalling the interpretation of RGA given in the book, the entry (1,1) in RGA(G(0)) corresponds to $\tilde{G}(0)\Big|_{K=0} \cdot \frac{1}{\tilde{G}(0)}\Big|_{K=\infty} = -3$.

3. (a) The minimal state space realization of $G(s)$ in controller canonical form is

$$A = \begin{bmatrix} -1.2 & -0.2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

   (b) Assuming

$$Q_1 = I_2, \quad Q_2 = 1, \quad R_1 = 1, \quad \text{and} \quad R_2 = 1$$

   and using `lqe`+`lqr` (or `lqg`) one gets

$$K_1 = \begin{bmatrix} 0.4690 \\ 0.3963 \end{bmatrix}, \qquad L_1 = \begin{bmatrix} 0.8198 & 0.8198 \end{bmatrix}$$

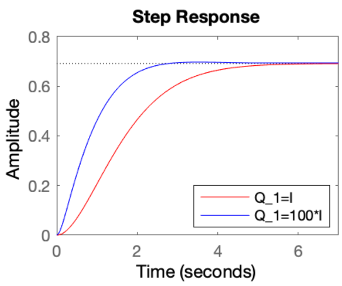
(c) To make the regulation faster it is enough to increase Q_1 (or to reduce Q_2). For instance, taking $Q_1 = 100 I_2$, one gets

$$K_2 = K_1, \qquad L_2 = \begin{bmatrix} 9.8020 & 9.8020 \end{bmatrix}$$

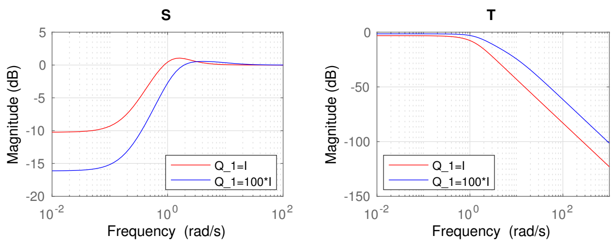
The two step responses are shown in Fig. 1(a). The blue one (corresponding to $Q_1 = 100 I_2$) is faster.

(d) S and T are plotted in Fig. 1(b). The blue system is less sensitive to system disturbances (because the input is allowed to be larger) but more sensitive to measurement noise (for the same reason).

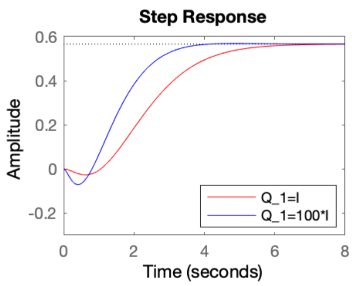
(e) The system $\tilde{G}(s)$ is non-minimum phase (zero in RHP), hence in the step response we expect that the output starts on the wrong direction. This is indeed what happens, see Fig. 1(c).



(a)



(b)



(c)

Figure 1: Ex. 3. (a): Step responses. (b): S and T for the two controllers. (c): Step response for the non-minimum phase system $\tilde{G}(s)$.

4. (a) Self-sustained oscillations are present when $\frac{-1}{Y_f(C)} = G(i\omega) \Leftrightarrow -(1+C^2) = G(i\omega) = \frac{iK(\omega^2-2)}{\omega(\omega^2+1)(\omega^2+4)} + \frac{-3K}{(\omega^2+1)(\omega^2+4)}$. The left-hand side is real and negative, with a maximum value of -1 when $C = 0$. Intersections can occur when $\mathrm{im}(G(i\omega)) = 0$, which corresponds to $\omega = \sqrt{2}$. Furthermore, in $\omega = \sqrt{2}$, $G(i\sqrt{2}) = \frac{-K}{6}$, and the condition forintersections becomes $\frac{-K}{6} < -1$, that is, $K > 6$.

(b) When oscillations occur, it is $-(1+C^2) = \frac{-K}{6} \Rightarrow C = \sqrt{\frac{K}{6} - 1}$.

(c) An increase in the amplitude C corresponds to a movement towards left on the curve $\frac{-1}{Y_f(C)}$. The system is therefore stable for large amplitudes and unstable for small amplitudes, meaning that the self-sustained oscillation is stable in amplitude.

5. (a) The only equilibrium point of the system is $x_{\text{eq}} = 0$.

(b) The Jacobian linearization of the system at $x_{\text{eq}} = 0$ gives $A = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ which has purely imaginary eigenvalues $\lambda_{1,2} = \pm i\sqrt{2}$. Hence the stability character of $x_{\text{eq}} = 0$ cannot be decided by the linearization. One can try Lyapunov functions, and one possible Lyapunov function is $V(s) = \frac{1}{2}(x_1^2 + 2x_2^2)$. This is positive definite and has derivative $\dot{V}(x) = -x_1^2 x_2^2$ which is negative semidefinite. Hence the equilibrium point is at least stable. To check that it is also asymptotically stable, one must consider the level sets of $V(x)$:

$$\mathcal{L} = \{x \text{ s.t. } \dot{V}(x) = 0\} = \{x \text{ s.t. either } x_1 = 0 \text{ or } x_2 = 0\}$$

and show that there are no trajectories of the system that lie in $\mathcal{L}$:

$$\begin{array}{lllll} x_1 = 0 \Rightarrow & \dot{x}_2 = 0 \Rightarrow & x_2 = \text{const} \Rightarrow & \dot{x}_1 = 2x_2 \neq 0 \Rightarrow & x_1 \text{ changes} \\ x_2 = 0 \Rightarrow & \dot{x}_1 = 0 \Rightarrow & x_1 = \text{const} \Rightarrow & \dot{x}_2 = -x_1 \neq 0 \Rightarrow & x_2 \text{ changes} \end{array}$$

Hence no other trajectory that the equilibrium point $x_{\text{eq}} = 0$ lies in $\mathcal{L}$, meaning that $x_{\text{eq}} = 0$ is asymptotically stable.

(c) We have already computed the Jacobian linearization matrix A. The input matrix is $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and the pair (A, B) is controllable, hence one can use Jacobian linearization to design a controller. For instance, $L = \begin{bmatrix} 3 & 0 \end{bmatrix}$ places the poles of the linearized closed loop $A - BL$ in -1, -2. The linear controller $u = -Lx$ is valid locally also for the nonlinear system.

(d) For the original nonlinear system, choosing a change of feedback $u = x_1 x_2^2 + v$ leads to a linear dynamics

$$\dot{x} = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v$$

which is controllable, and for which the same linear feedback $v = -Lx$ used above is valid. Hence the complete nonlinear feedback is $u = x_1 x_2^2 - Lx$.

(e) Any of the above would do. Notice that since $x_{\text{eq}} = 0$ is already asymptotically stable, even $u = 0$ would be a correct answer...