Solution for TSRT09 Control Theory, 2023-03-24

1. (a) See section 7.4 of the book.

   (b) (i) The controllability matrix

$$S = \begin{bmatrix} 1 & 0 & -2 & 0 & 4 & 0 & -8 & 0 \\ 0 & 2 & 0 & -2 & 0 & 2 & 0 & -2 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 2 & 0 & -2 & 0 & 2 \end{bmatrix}$$

has $\text{rank}(S) = 3$, hence the system is not controllable. The controllable subspace (i.e., $\text{range}(S)$) is for instance generated by

$$\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}\right)$$

   (ii) The observability matrix

$$O = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -8 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

has $\text{rank}(O) = 3$, hence the system is not observable. The non-observable subspace (i.e., $\ker(O)$) is for instance generated by

$$\text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}\right)$$

   (iii) Since all eigenvalues of A are in LHP, the system is both stabilizable and detectable.

   (iv) Since the system is neither controllable nor observable the realization is not minimal.

   (v) $G(s)$ is straightforwardly computed as

$$G = \begin{bmatrix} \frac{1}{s+2} & \frac{2}{s+1} \\ \frac{1}{s+1} & \frac{-2}{s+1} \end{bmatrix}$$

of pole polynomial $p(s) = (s+1)^2(s+2)$ and zero polynomial $z(s) = 2s+3$. Hence the poles are $s = -1$ of multiplicity 2, and $s = -2$ of multiplicity 1. The zero is $s = -1.5$.

2. (a) From $x = F(s)v_1 = \frac{1}{s+2}v_1$, the spectrum is $\phi_x(\omega) = F(i\omega)r_1F^*(i\omega) = \frac{r_1}{\omega^2+4}$. The variance of x, $\pi_x = E[x^2]$, can be computed from the Lyapunov equation (Theorem 5.3 of the book)

$$-2\pi_x - 2\pi_x + r_1 = 0 \quad \Longrightarrow \quad \pi_x = \frac{r_1}{4}$$

(b) The observer equation can be rewritten as $\dot{\hat{x}} = -(2+K)\hat{x} + Ky$. Laplace transforming: $G(s) = \frac{K}{s+2+K}$.

(c) Similarly, $\dot{\tilde{x}} = -(2+K)\tilde{x} + v_1 - Kv_2$. Laplace transforming:

$$H_1(s) = \frac{1}{s+2+K}, \qquad H_2(s) = -\frac{K}{s+2+K}$$

The associated variances can be computed solving Lyapunov equations, similarly to (a):

$$E\left[\tilde{x}_1^2\right] = \frac{r_1}{2(2+K)}, \qquad E\left[\tilde{x}_2^2\right] = \frac{r_2K^2}{2(2+K)}$$

(d) We need to find the minimum of the total variance $V(K) = \frac{r_1+r_2K^2}{2(2+K)}$. If $2+K>0$ (i.e., when the observer is stable), then $V(K)>0$, and its minimum in K can be found computing where the derivative vanishes:

$$\frac{\partial V(K)}{\partial K} = \frac{2r_2K^2 + 8r_2K - 2r_1}{4(2+K)^2} = 0 \quad \Longrightarrow \quad K_{1,2} = -2 \pm \sqrt{4 + \frac{r_1}{r_2}}$$

To guarantee stability of the observer (i.e., $-(2+K)<0$) we need to chose the solution with + sign: $K_{\text{min}} = -2 + \sqrt{4 + \frac{r_1}{r_2}}$.

(e) The Kalman filter is obtained choosing $K_{\text{kalman}} = p/r_2$, where p solves the Riccati equation:

$$-2p - 2p - \frac{p^2}{r_2} + r_1 = 0 \quad \Longrightarrow \quad p_{1,2} = -2r_2 \pm \sqrt{4r_2^2 + r_1r_2}$$

Also here, since p must be positive definite, we must choose the solution with + sign: $K_{\text{kalman}} = -2 + \sqrt{4 + \frac{r_1}{r_2}}$. Not surprisingly, it is $K_{\text{kalman}} = K_{\text{min}}$.

3. (a) The extended system is

$$G_e = \begin{bmatrix} 0 & W_u \\ 0 & W_TG \\ W_S & W_SG \\ 1 & G \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{(s+1)(s+2)}{(s+0.1)(s+3)(s+10)} \\ \frac{10}{s+1} & \frac{10}{(s+0.1)(s+3)} \\ 1 & \frac{s+1}{(s+0.1)(s+3)} \end{bmatrix}$$

(b) For $\gamma = 3.0644$, the $\mathcal{H}_\infty$ controller is

$$F_y = \frac{294.19(s+10)(s+3)(s+0.1)}{(s+99.05)(s+10.03)(s+1.301)(s+1)}$$

(c) The $\mathcal{H}_2$ controller is

$$F_y = \frac{6.0952(s+10)(s+3)(s+0.1)}{(s+10.06)(s+4.505)(s+1)(s+0.8632)}$$

(d) S, T and G_{wu} for the two designs are shown in Fig. 1. As can be seen, the $\mathcal{H}_2$ loopshaping corresponds to bigger sensitivity at low frequencies, compensated by a lower complementary sensitivity at higher frequencies. Overall the $\mathcal{H}_\infty$ closed loop transfer function G_c has larger bandwidth than the $\mathcal{H}_2$ design, and a higher static gain. Correspondingly, the step response of the close loop system is prompter for the $\mathcal{H}_\infty$ than for the $\mathcal{H}_2$ design.

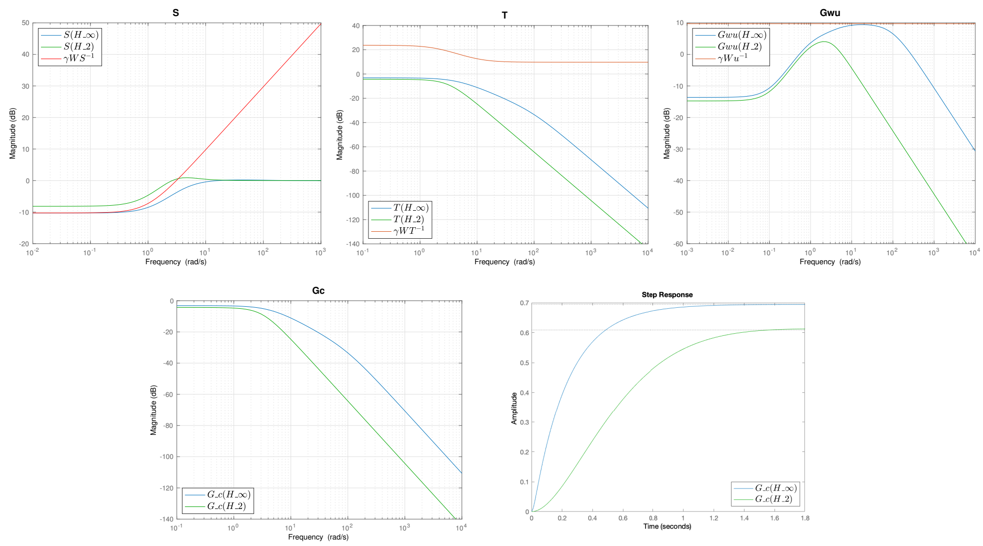
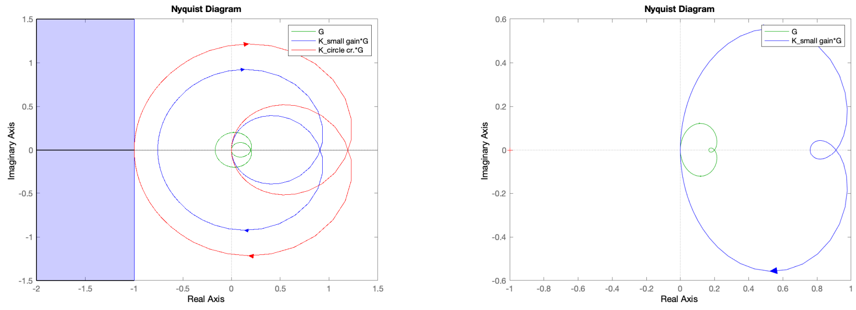


Figure 1: $\mathcal{H}_2$ (green) and $\mathcal{H}_\infty$ (blue) design of Ex. 3.

4. (a) Consider the transfer function in Eq. (3) of the text. Since $\tanh(v) \leq v$ for all $v \geq 0$ (and $\tanh(v) \geq v$ for all $v \leq 0$), the nonlinearity has unit gain: $\|\tanh(v)\| = 1$. Hence for the small gain theorem, $\|KG\|_\infty \|\tanh(v)\| < 1$ simply means $K\|G\|_\infty < 1$. Since $\|G\|_\infty = 0.2193$ (i.e. -13.1805 dB), it must be $K < 4.5606$.

(b) The circle associated to $\tanh(v)$ has $k_1 = 0$ and $k_2 = 1$, hence it has an infinite radius and it is passing through -1, i.e., the region to avoid is the half plane to the left of -1, see blue area in Fig. 2(a). For the circle criterion to be satisfied, the "loop gain" $KG(i\omega)$ must be to the right of this region. The Nyquist curve of $G(i\omega)$ is also shown in green in Fig. 2(a). Computing the rightmost value of $G(i\omega)$ (i.e., $\eta = \min_\omega(\text{Real}(G(i\omega)))$, this value is $\eta = 0.1667$, meaning that if $K < 1/\eta = 6$ the circle criterion is satisfied, see red curve in Fig. 2(a). As expected, the circle criterion is less conservative than the small gain theorem.

(c) The transfer function of Eq. (4) is the minimum phase version of the one in Eq. (3). Hence the two have the same $\|G\|_\infty = 0.2193$ (minimum and nonminimum phase TF have the same amplitude), meaning that the small gain condition on K is the



(a) (b)

Figure 2: Ex. 4. (a): Nyquist curve for $G(s)$ and $KG(s)$ in Eq. (3) of the text. (b): Nyquist curve for $G(s)$ and $KG(s)$ in Eq. (4).

same. From the Nyquist plot of the new TF, shown in Fig. 2(b) in green, it can be observed that the amplitude margin is ∞ (the Nyquist curve is all in the RHP), meaning that the closed loop system is stable for all $K > 0$ (the "loop gain" $KG(i\omega)$ remains in the LHP for all $K > 0$).

(d) As already mentioned, the TF of Eq. (4) is the minimum phase version of the one in Eq. (3). Nonminimum phase systems are more stability-critical, and this is just one example of such criticality.

5. (a) (i) The Jacobian linearization is

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

which has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 0$, hence it is an undecidable case.

(ii) $V(x) > 0$ (p.d.) and

$$\dot{V}(x) = -4x_1^2(1 + 2x_1^2) - 4x_2^4 < 0$$

hence this can be used.

(iii) $V(x)$ is not a p.d. function: $V(\bar{x}) = 0$ when $\bar{x} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$, hence this cannot be used.

(iv) If $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then we get the $V(x) = x_1^2 + x_2^2$ used in (ii), hence this is OK.

(v) $V(x) = x_1^2 + x_2^2$ is radially unbounded, hence the asymptotic stability of the origin is global.

(b) The Jacobian linearization is now

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$$

which has eigenvalue $\lambda = -2$ of multiplicity 2, meaning that local asymptotic stability holds. Also in this case asymptotic stability is global: choosing $V(s) = x_1^2 + x_2^2$, one gets

$$\dot{V}(x) = -4x_1^2 + 2x_1x_2 - 8x_1^4 - 4x_2^2 - 4x_2^4 = -(x_1 - x_2)^2 - (3x_1^2 + 3x_2^2 + 8x_1^4 + 4x_2^4) < 0$$

The matrix A is already in Jordan normal form, meaning that the linearization has a single eigenvector. A sketch of the phase portrait is shown in Fig. 3.

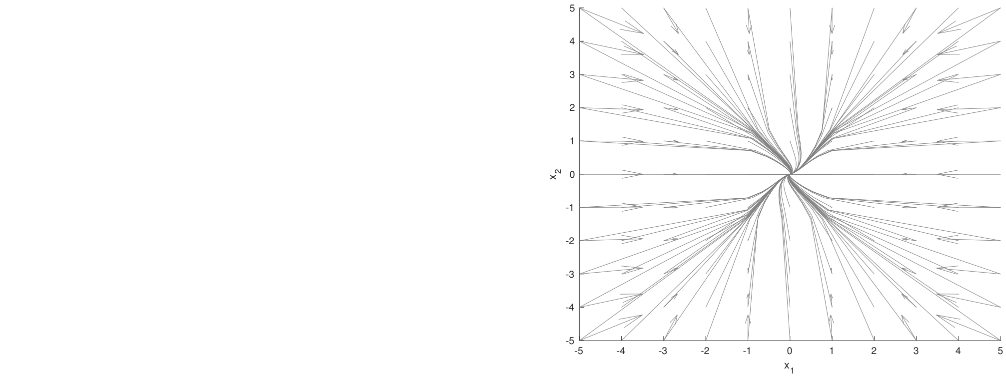


Figure 3: Phase portrait of the system of Ex. 5.