Solution for TSRT09 Control Theory, 2022-06-08

1. (a) A system is controllable if all its poles can be allocated arbitrarily by a state feedback; it is stabilizable if all its unstable poles (but not necessarily the stable ones) can be allocated arbitrarily by state feedback. The PBH test can be used for checking both properties.

   (b) The case $p > z$ is the most difficult, as according to eq. (7.31) and (7.35) in the course book, the cross-over frequency (and hence the bandwidth of the closed loop system) must be less than z and larger than p. When $p > z$, this results in poor behavior for the sensitivity function and for the complementary sensitivity function.

   (c) The transfer function (iii) has poles in $-0.2 \pm i3.995$ meaning that it has a resonance peak around 4 rad/sec.

   (d) Let $G(s) = \frac{1}{s+2}$ and $G_d(s) = \frac{2}{s+3}$. For perfect disturbance cancellation it must be $u = -G^{-1}G_d v$. If the disturbance is sinusoidal, $v(t) = 2\sin(3t)$, also u becomes sinusoidal, with a gain factor of $|G^{-1}(i3)G_d(i3)|$. We then get the constraint $u_0 \geq |G^{-1}(i3)G_d(i3)|2 = \frac{2}{3}\sqrt{26} \approx 3.40$.

2. (a) The transfer function of the system is

$$G(s) = C(sI - A)^{-1}B = \begin{bmatrix} \frac{2s+3}{(s+2)(s+1)} & \frac{1}{s+2} \\ \frac{1}{s+2} & \frac{2s+5}{(s+2)(s+3)} \end{bmatrix}$$

   meaning that the pole polynomial is $p(s) = (s+1)(s+2)(s+3)$. The poles are -1, -2, -3, and the zero is -2.

   (b) $\|G\|_\infty = 1.7676$ (or 4.94 dB).

   (c) The RGA at 0 frequency is

$$\text{RGA}(G(0)) = G(0) \odot (G^{-1}(0))^T = \begin{bmatrix} 1.2500 & -0.2500 \\ -0.2500 & 1.2500 \end{bmatrix}$$

   (d) The sensitivity function is

$$\begin{aligned} S(i\omega) &= (I + GF)^{-1} \\ &= \frac{1}{s^3 + 86s^2 + 1531s + 2686} \begin{bmatrix} s^3 + 46s^2 + 151s + 106 & -20s^2 - 80s - 60 \\ -20s^2 - 80s - 60 & s^3 + 46s^2 + 191s + 186 \end{bmatrix} \end{aligned}$$

   (e) S is the transfer function from disturbance w to control error (here equal to $-y$), hence looking at the singular values of S, shown in Figure 1, one can see that the gain from w to control error is < -20dB for $\omega < 1\,\text{rad/sec}$.

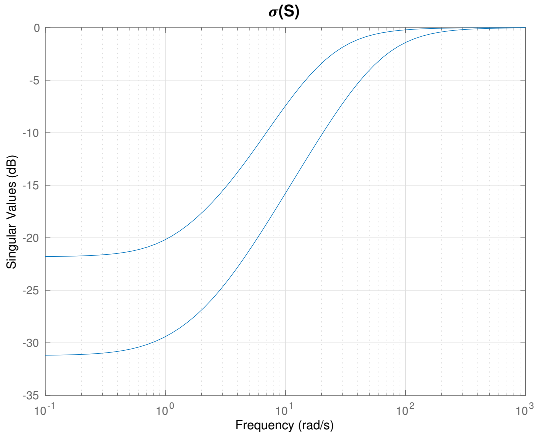

Figure 1: Singular values for S in Exercise 2.

(f) The disturbance is constant, hence we have to look at $\omega = 0$. Computing the "best" ratio w_1/w_2 corresponds to computing the eigenvector of the largest singular value of $S(0)$ (i.e., eigenvalue of $\sqrt{S^*(0)S(0)}$). Since $\sigma(S(0)) = \{0.0275, 0.0812\}$, we have to look for the eigenvector associated to 0.0812, which is $w = [-0.4719\ 0.8817]^T$, meaning that the sought ratio is $w_1/w_2 = -0.5352$.

3. Let

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} = \begin{bmatrix} 48 & 0 \\ 0 & 1 \end{bmatrix}$$

(a) When the observer gain is $K_1 = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$, denoting $\hat{x}_1$ the state observer, then

$$\dot{\hat{x}}_1 = (A - K_1 C)\hat{x}_1 + K_1 y$$

and the eigenvalues of $A - K_1 C$ are both equal to -2.

(b) If $\tilde{x}_1 = x - \hat{x}_1$, the estimation error ODE is

$$\dot{\tilde{x}}_1 = (A - K_1 C)\tilde{x}_1 + \begin{bmatrix} N & -K_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

The associated covariance matrix $P_1 = E\tilde{x}_1\tilde{x}_1^T$ can be obtained by solving a Lyapunov equation (in matlab: `P1=lyap(A-K1*C, [N -K1]*R*[N'; -K1'])`:

$$P_1 = \begin{bmatrix} 2.2812 & 4.8438 \\ 4.8438 & 19.6562 \end{bmatrix}$$

(c) The Kalman filter (here denoted K_2, of error covariance P_2) can be computed in matlab using `[K2 P2,pole2]=lqe(A,N,C,R1,R2,0)`, which gives

$$K_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 2 & 4 \\ 4 & 16 \end{bmatrix} \qquad \text{and poles} = -2 \pm 1.732i$$

(d) The variance of the estimation error, $\Sigma_i = E[\tilde{x}_i^T \tilde{x}_i]$, is nothing but the trace of P_i, hence

$$\Sigma_1 = \text{trace}(P_1) = 21.9375 > \Sigma_2 = \text{trace}(P_2) = 18$$

as expected.

4. (a) From Chapter 14 in the book, the describing function for the relay with hysteresis is

$$Y_f(C) = \frac{4}{\pi C}\left(\sqrt{1-\frac{1}{4C^2}} - \frac{i}{2C}\right), \qquad C \geq 0.5$$
$$-1/Y_f(C) = -\frac{\pi C}{4}\sqrt{1-\frac{1}{4C^2}} - i\frac{\pi}{8}.$$

The frequency function for the system is

$$G(i\omega) = \frac{10}{i\omega(1+i\omega)} = \frac{-10(\omega+i)}{\omega(\omega^2+1)}.$$

The intersection is obtained by solving the system of equations (in ω and C): $G(i\omega) = -1/Y_f(C)$;

$$\text{Imaginary part:} \quad \frac{10}{\omega(\omega^2+1)} = \frac{\pi}{8}$$
$$\text{Real part:} \quad \frac{10}{\omega^2+1} = \frac{\pi}{8}\sqrt{4C^2-1}.$$

The solution is $\omega_o = 2.8288$ and $C_o = 1.5002$. This can be checked also graphically, see Fig. 2.

(b) The curve $-\frac{1}{Y_f(C)}$ grows moving from right to left.

$C < C_o \implies -1/Y_f(C)$ is encircled by $G(i\omega) \implies$ oscillation grows

$C > C_o \implies -1/Y_f(C)$ is not encircled by $G(i\omega) \implies$ oscillation decreases

Summing up: the oscillation is stable in amplitude.

5. (a) Choosing state space variables $x_1 = \int e$ and $x_2 = y$, together with $u = \text{sat}(x_1 + e)$ and $e = -y$ (since $r = 0$), one gets the nonlinear state space model

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = \text{sat}(x_1 - x_2)$$

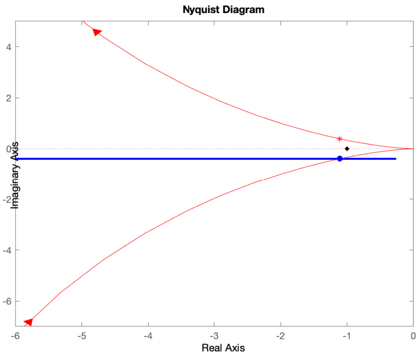

Figure 2: Exercise 4. In red: $G(i\omega)$, in blue $-\frac{1}{Y_f(C)}$. Solid blue point: intersection.

which corresponds to a system switching between 3 different right hand sides in 3 different regions of the state space:

A: for $|x_1 - x_2| \leq 1$, it is

$$\dot{x} = Ax, \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

B: for $x_1 - x_2 < -1$ (i.e., $x_2 > x_1 + 1$) it is

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = -1$$

C: for $x_1 - x_2 > 1$ (i.e., $x_2 < x_1 - 1$) it is

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = 1$$

Only the first region admits an equilibrium point corresponding to $x = 0$. Since the eigenvalues of A are $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, the equilibrium point is a stable focus.
Phase portrait:

A: The trajectories spiral down to the origin;

B: It is $\frac{\dot{x}_1}{\dot{x}_2} = \frac{dx_1}{dx_2} = x_2$, whose solution $x_1 = \frac{1}{2}x_2^2 + c$ is a parabola. Furthermore, x_2 keeps decreasing, while x_1 increases for $x_2 < 0$ and decreases otherwise;

C: The solution in this case is $x_1 = -\frac{1}{2}x_2^2 + c$, and x_2 keeps growing, while x_1 decreases if $x_2 > 0$ and increases otherwise.

The overall phase portrait can be seen in Fig. 3(a).

(b) Similarly to part (a), there are 3 regions divided by the lines $x_1 - x_2 = -1$ and $x_1 - x_2 = 1$, and characterized by the 3 systems

A: for $|x_1 - x_2| \leq 1$, same as before

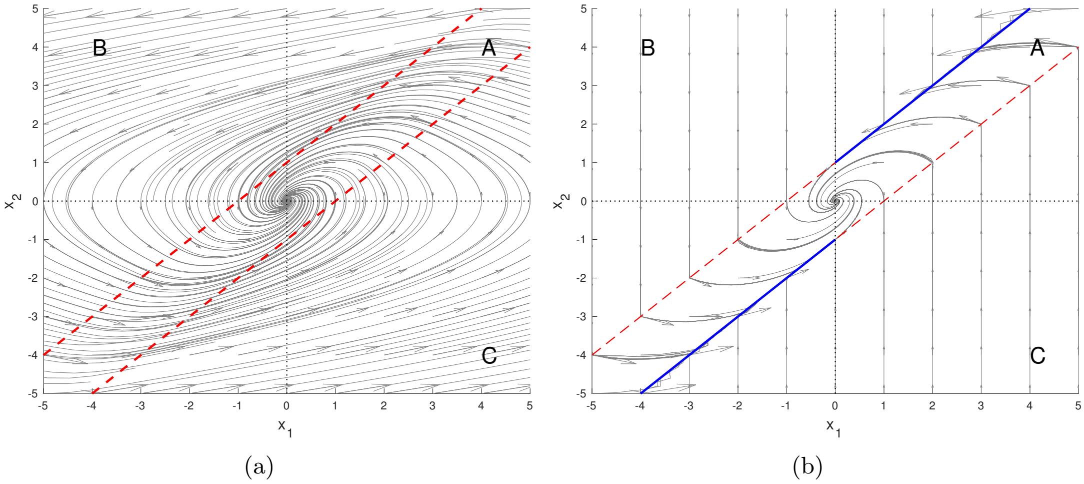

Figure 3: Phase portraits for Exercise 5.

B: for $x_1 - x_2 < -1$

$$\dot{x}_1 = 0$$
$$\dot{x}_2 = -1$$

C: for $x_1 - x_2 > 1$

$$\dot{x}_1 = 0$$
$$\dot{x}_2 = 1$$

The new phase portrait is in Figure 3(b).

To show chattering behavior, let us look at what happens as the trajectories cross the dividing lines. Let us consider first $x_1 - x_2 = -1$, for $x_1 > 0$ (and $x_2 > 1$). A trajectory coming from the region **A** has $\dot{x}_1 = -x_2 < -1$ and $\dot{x}_2 = x_1 - x_2 = -1$, which implies that it is pushed out of the region **A** towards **B**. But then in **B** the system becomes $\dot{x}_1 = 0$ and $\dot{x}_2 = -1$ i.e., the trajectory is forced back into **A**. This is a chattering behavior: the value of $\dot{x}_1$ switches very quickly between 0 and $-x_2 < -1$. Something similar does not happen when $x_1 < 0$ (and $x_2 < 1$) where instead the trajectories enter from **B** to **A** and continue in **A** afterwards, see Figure 3(b).

A specular behavior occurs in the other dividing line $x_1 - x_2 = 1$.

Overall, chattering occurs on the blue lines shown in Figure 3(b). What happens in practice in a simulator is that the trajectory slides along the blue line towards the origin until it reaches the vertical axis. After that it enters the region **A** definitively and spirals towards the origin.