Optimal Control, Lecture 9: The Pontryagin maximum principle (PMP)

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Contents

Optimal Control Problem

$$\begin{array}{ll} \text{minimize} & \phi\left(x(T)\right) + \int_0^T f\left(t, x(t), u(t)\right) dt \\ \text{subject to} & \dot{x}(t) = F\left(t, x(t), u(t)\right) \end{array} \qquad (1)$$

with variables x and u, $x(0) = x_0$ given, and $F : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$, $f : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ and $\phi : \mathbf{R}^n \to \mathbf{R}$ continuously differentiable.

Assume $x \in \mathcal{C}^n(0, T)$ and $u \in \mathcal{C}^m(0, T)$

Optimal u denote by $u^\star$.

Assume that the corresponding solution $x^\star$ to the differential equation is unique, and that in case $u^\star$ is perturbed with a small amount, the corresponding perturbation of $x^\star$ is also small.

Lagrangian Functional

Define the *Lagrangian* $L : \mathcal{C}^m(0,T) \to$ **R** as

$$\begin{aligned} L[u] &= \phi\left(x(T)\right) + \int_0^T \Big(f\left(t, x(t), u(t)\right) \\ &+ \lambda(t)^T \left(F\left(t, x(t), u(t)\right) - \dot{x}(t)\right)\Big) dt \\ &= \phi\left(x(T)\right) + \int_0^T \left(H\left(t, x(t), u(t), \lambda(t)\right) - \lambda(t)^T \dot{x}(t)\right) dt, \end{aligned}$$

where H : **R** $\times$ **R**^{n} $\times$ **R**^{m} $\times$ **R**^{n} $\to$ **R** is the *Hamiltonian* defined as

$$H\left(t, x, u, \lambda(t)\right) = f(t, x, u) + \lambda^T F(t, x, u)$$

Perturbations

Make perturbation $u = u^\star + \delta u$ of $u^\star$, where δu is small, *i.e.*, $\|\delta u\| < \epsilon$.

Corresponding perturbed trajectory x, which is the solution of differential equation for u differs from the original solution $x^\star$ with the quantity $\delta x = x - x^\star$, which by our assumption is small, *i.e.*, $\|\delta x\|$ can be made as small as we like by taking ϵ sufficiently small.

Increment of L

$$
\begin{aligned}
\Delta L[\delta u] = L[u^\star + \delta u] - L[u^\star] &= \phi\left(x^\star(T) + \delta x(T)\right) - \phi\left(x^\star(T)\right) \\
&+ \int_0^T \Big(H\left(t, x^\star(t) + \delta x(t), u^\star(t) + \delta u(t), \lambda(t)\right) \\
&- H\left(t, x^\star(t), u^\star(t), \lambda(t)\right) \Big) dt \\
&- \int_0^T \lambda(t)^T \frac{d}{dt}\left(x^\star(t) + \delta x(t)\right) dt + \int_0^T \lambda(t)^T \frac{d}{dt} x^\star(t)) dt.
\end{aligned}
$$

First Variation

From Taylor series expansion

$$\delta L = \frac{\partial \phi}{\partial x^T} \delta x(T) + \int_0^T \left(\frac{\partial H}{\partial x^T} \delta x(t) + \frac{\partial H}{\partial u^T} \delta u(t) - \lambda(t)^T \frac{d\delta x(t)}{dt} \right) dt.$$

which is clearly linear in $\delta u(t)$, $\delta x(t)$, and its derivative.

The assumption on small perturbations of δu resulting in small perturbations of δx is necessary. Otherwise the remainder term does not converge to zero as $\|\delta u\| \to 0$.

Integration by parts

$$\delta L = \frac{\partial \phi}{\partial x^T}\delta x(T) + \int_0^T \left(\frac{\partial H}{\partial x^T}\delta x(t) + \frac{\partial H}{\partial u^T}\delta u(t) + \frac{d\lambda(t)^T}{dt}\delta x(t)\right) dt$$
$$- \left[\lambda(t)^T \delta x(t)\right]_0^T$$

Since initial value $x(0)$ is given it follows that $\delta x(0) = 0$, and hence

$$\delta L = \left(\frac{\partial \phi}{\partial x^T} - \lambda(T)^T\right)\delta x(T)$$
$$+ \int_0^T \left(\frac{\partial H}{\partial x^T}\delta x(t) + \frac{\partial H}{\partial u^T}\delta u(t) + \frac{d\lambda(t)^T}{dt}\delta x(t)\right) dt$$

Adjoint Equations

Let λ satisfy the *adjoint equations*

$$\dot{\lambda}(t) = -\frac{\partial H(t, x^\star(t), u^\star(t), \lambda(t))}{\partial x}, \qquad \lambda(T) = \frac{\partial \phi(x^\star(T))}{\partial x}$$

This is a linear time-varying differential equation and hence it has a solution under mild conditions on H.

E.g. if $\frac{\partial f(t,x^\star(t),u^\star(t))}{\partial x}$ and $\frac{\partial F(t,x^\star(t),u^\star(t))}{\partial x}$ are bounded functions of t on $[0, T]$.

As a Result

$$\delta L = \int_0^T \frac{\partial H}{\partial u^T} \delta u(t) dt.$$

From the du Bois-Raymond lemma it then follows that

$$\frac{\partial H(t, x^\star(t), u^\star(t), \lambda(t))}{\partial u^T} = 0$$

in order for the first variation to vanish.

Further Necessary Condition

For differentiable $u^\star$ it holds that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x^T}\dot{x}^\star + \frac{\partial H}{\partial u^T}\dot{u}^\star + \frac{\partial H}{\partial \lambda^T}\dot{\lambda} = \frac{\partial H}{\partial t} + \left(\frac{\partial H}{\partial x} + \dot{\lambda}\right)^T F = \frac{\partial H}{\partial t}$$

where all functions are evaluated for $(x^\star, u^\star)$.

It is possible to show that the result holds also for piecewise continuous $u^\star$.

In case H does not explicitly depend on t, called an *autonomous optimal control problem*, it holds that H is a constant independent of t.

Pontryagin Maximum Principle (PMP)

Given optimal $u^\star$ and $x^\star$ for (1), there exist an *adjoint variable* λ such that

$$\dot{\lambda}(t) = -\frac{\partial H(t, x^\star(t), u^\star(t), \lambda(t))}{\partial x}, \qquad \lambda(T) = \frac{\partial \phi(x^\star(T))}{\partial x} \tag{2a}$$

$$\frac{\partial H(t, x^\star(t), u^\star(t), \lambda(t))}{\partial u^T} = 0 \tag{2b}$$

$$\frac{dH(t, x^\star(t), u^\star(t), \lambda(t))}{dt} = \frac{\partial H(t, x^\star(t), u^\star(t), \lambda(t))}{\partial t}. \tag{2c}$$

Note: These necessary conditions for optimality hold for any extremum.

Sufficient Conditions for Optimality

Based on the second variation of the Lagrangian L. Let $(\bar{u}, \bar{x})$ and λ satisfy

$$
\begin{aligned}
&\dot{x}(t) = F(t, \bar{x}(t), \bar{u}(t)), \quad x(0) = x_0 \\
&\dot{\lambda}(t) = -\frac{\partial H(t, \bar{x}(t), \bar{u}(t), \lambda(t))}{\partial x}, \qquad \lambda(T) = \frac{\partial \phi(\bar{x}(T))}{\partial x} \\
&\frac{\partial H(t, \bar{x}(t), \bar{u}(t), \lambda(t))}{\partial u} = 0 \\
&\frac{dH(t, \bar{x}(t), \bar{u}(t), \lambda(t))}{dt} = \frac{\partial H(t, \bar{x}(t), \bar{u}(t), \lambda(t))}{\partial t} \\
&\frac{d^2 \phi(\bar{x}(T))}{dx dx^T} \succeq 0, \quad \frac{\partial^2 H(t, \bar{x}(t), \bar{u}(t), \lambda(t))}{\partial z \partial z^T} \succeq 0 \\
&\frac{\partial^2 H(t, \bar{x}(t), \bar{u}(t), \lambda(t))}{\partial u \partial u^T} \succ 0
\end{aligned}
$$

where $z = (x, u)$. Then $(\bar{u}, \bar{x})$ is a local minimum of (1).

Linear Quadratic (LQ) Control

$$
\begin{array}{ll}
\text{minimize} & \frac{1}{2}x(T)^T Q_0 x(T) + \frac{1}{2}\int_0^T \big(x(t)^T Q x(t) + u(t)^T R u(t)\big)\,dt \\
\text{subject to} & \dot{x}(t) = Ax(t) + Bu(t)
\end{array}
$$

with variables x and u for given initial value $x(0) = x_0$. The Hamiltonian is given by

$$H = \frac{1}{2}\left(x^T Q x + u^T R u\right) + \lambda^T (Ax + Bu).$$

We realize that the adjoint equations are

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -Qx - A^T\lambda, \quad \lambda(T) = Q_0 x(T).$$

From the PMP we have that

$$\frac{\partial H}{\partial u} = Ru + B^T\lambda = 0$$

If we assume that $R \succ 0$, *i.e.*, positive definite, we have that $u = -R^{-1}B^T\lambda$.

Two Point Boundary Value Problem

If we insert this into the differential equations for x and λ we obtain

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ \lambda(T) \end{bmatrix} = \begin{bmatrix} x_0 \\ Q_0 x(T) \end{bmatrix}.$$

Remaining part on white board.

Riccati Equation

Let P solve

$$\dot{P} + A^T P + PA + Q - PBR^{-1}B^T P = 0$$

with boundary condition

$$P(T) = Q_0$$

Then $\lambda(t) = P(t)x(t)$, and hence

$$u(t) = -R^{-1}B^T \lambda(t) = -R^{-1}B^T P(t)x(t)$$

Proof on white board.

Euler–Lagrange Equations

Consider case when $\dot{x} = F(t, x, u) = u$ Then $H = f + \lambda^T u$.
Hence (2a–2b) become

$$\dot{\lambda} = -\frac{\partial f}{\partial x}$$
$$\frac{\partial f}{\partial u} + \lambda = 0,$$

implying

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) - \frac{\partial f}{\partial x} = 0, \qquad (3)$$

Necessary condition for an optimal $x^\star$ of the problem

$$\text{minimize} \int_0^T f\left(t, x(t), \dot{x}(t)\right) dt$$

with variable x.

Conservative Mechanical Systems

State x contains positions and angels, and $\dot{x}$ is called generalized velocity.

Potential energy $V(x)$ and kinetic energy $T(x, \dot{x})$, where $T(x, \dot{x}) = \dot{x}^T A(x)\dot{x}$ for some symmetric matrix $A(x)$.

Let the Lagrangian of mechanics be $f(x, \dot{x}) = T(x, \dot{x}) - V(x)$.

State should be an extremum of

$$\int_0^T f(x(t), \dot{x}(t))dt.$$

i.e. x should solve Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0.$$

Conservative Mechanical Systems ctd.

The Hamiltonian is

$$
\begin{aligned}
H &= f + \lambda^T \dot{x} = T - V + \lambda^T \dot{x} = T - V - \frac{\partial f}{\partial \dot{x}} \dot{x} \\
&= \dot{x}^T A \dot{x} - V - 2\dot{x} A \dot{x} = -T - V
\end{aligned}
$$

Since the system is autonomous, the Hamiltonian should be a constant, and hence we have proven that the sum of potential and kinetic energy is constant for conservative mechanical systems.

Beltrami's Identity

Assume that f does not depend explicitly on t. From Euler-Lagrange equations

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial \dot{x}}\ddot{x} = \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\dot{x}\right),$$

where the first equality follows by assumption and the second equality by the chain rule. Hence

$$\frac{d}{dt}\left(f - \frac{\partial f}{\partial \dot{x}}\dot{x}\right) = 0,$$

from which it follows that

$$f - \frac{\partial f}{\partial \dot{x}}\dot{x} = C,$$

where C is a constant. This is Beltrami's identity.

Chain Example

Consider a chain hanging from two given points and determine the shape by minimizing the total potential energy of the chain.

A differential piece of the chain, of length ds has mass $dm = \rho ds$, where ρ is the mass density of the chain. Here $ds = \sqrt{1 + \dot{x}^2} dt$, where $x(t)$ is the position of the differential segment.

The potential energy of segment is $g\rho x(t) ds$ and total potential energy is

$$\int_0^T g\rho x \sqrt{1 + \dot{x}^2} dt,$$

Minimize integral subject to $x(0) = x_0$ and $x(T) = x_T$.

Chain Example ctd.

Beltrami's identity gives

$$g\rho x\sqrt{1+\dot{x}^2} - g\rho x\frac{\dot{x}^2}{\sqrt{1+\dot{x}^2}} = C.$$

We let $a = C/(g\rho)$ and obtain by multiplying with the square root

$$x = a\sqrt{1+\dot{x}^2}$$

or equivalently

$$\left(\frac{x}{a}\right)^2 - \dot{x}^2 = 1.$$

Then

$$x(t) = a\cosh\left(\frac{t-t_0}{a}\right)$$

satisfies the equation for any t_0. The boundary conditions give

$$a\cosh\left(\frac{-t_0}{a}\right) = x_0, \quad a\cosh\left(\frac{T-t_0}{a}\right) = x_T,$$

from which a and t_0 can be determined.