Optimal Control, Lecture 4: Model Predictive Control (MPC)

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Contents

Optimal Control Problem

$$J^\star(x_0) = \text{minimize } J = \sum_{k=0}^{\infty} f(x_k, u_k) \tag{1}$$
$$\text{subject to } x_{k+1} = F(x_k, u_k)$$
$$u_k \in U, \quad x_k \in X$$

where $x_0 \in X$ is given.

Bellman Equation

Assume $0 \in X$, $0 \in U$, $F(0,0) = 0$, $f(0,0) = 0$ and that f is strictly positive definite and quadratically bounded. If there exists a strictly positive definite function V that satisfies the Bellman equation

$$V(x) = \min_{u \in U,\ F(x,u) \in X} \{f(x,u) + V(F(x,u))\} \tag{2}$$

then for the optimal control problem in (1) it holds that

(a) $V(x) = J^\star(x)$

(b) $u^\star = \mu(x) = \operatorname{argmin}_{u \in U,\ F(x,u) \in X} \{f(x,u) + V(F(x,u))\}$ is an optimal feedback control that results in a globally convergent closed loop system.

Approximation of Value Function

In general difficult to solve Bellman equation to get V.

Guess approximation $\hat{V}$, which gives approximate feedback

$$u = \mu(x) = \underset{u \in U,\ F(x,u) \in X}{\operatorname{argmin}} \left\{ f(x,u) + \hat{V}(F(x,u)) \right\} \tag{3}$$

One Time-Step Horizon Problem

$$\begin{aligned}\text{minimize } & \hat{V}(x_{k+1}) + f(x_k, u_k) \\ \text{subject to } & x_{k+1} = F(x_k, u_k) \\ & u_k \in U, \quad x_{k+1} \in X\end{aligned}$$

where $x_k \in \mathcal{X}$ is given.

- Can be solved as finite-dimensional optimization problem
- Open loop solution
- Repeated on-line for every k results in feedback
- Called “greedy control” when $\hat{V}(x) = 0$
  - Often instability
  - Often too large value of J

Finite Time Horizon Approximation

$$\begin{aligned} \text{minimize } & J_N = \sum_{k=0}^{N-1} f(x_k, u_k) \\ \text{subject to } & x_{k+1} = F(x_k, u_k) \\ & u_k \in U, \quad x_k \in X, \quad x_N = 0 \end{aligned} \tag{4}$$

where $x_0 \in X$ is given.

- Large N implies near optimal solution
- $x_N = 0$ and $u_k = 0, k \geq N$ implies $x_k = 0, \ \forall k \geq N$, i.e. stability
- Can be solved as finite-dimensional optimization problem
- Open loop solution
- Repeated on-line for every k results in feedback (more later)

Finite-Dimensional Optimization

Let

$$z = \begin{bmatrix} x_1^T & \cdots & x_N^T & u_0^T & \cdots & u_{N-1}^T \end{bmatrix}^T$$

and define $f_0(z) = \sum_{k=0}^{N-1} f(x_k, u_k)$,

$$g(z) = \begin{bmatrix} f(x_0, u_0) - x_1 \\ \vdots \\ f(x_{N-1}, u_{N-1}) - x_N \\ x_N \end{bmatrix}$$

and the function $h : \mathbf{R}^{N(m+n)} \to \mathbf{R}^{Np}$ such that $h(z) \leq 0$ is equivalent to $u_k \in U$ for $k = 0, \ldots, N-1$ and $x_k \in X$, for $k = 1, \ldots, N$. Then the optimal control problem in (4) is equivalent to

$$\text{minimize } f_0(z) \text{ subject to } g(z) = 0,\ h(z) \leq 0$$

Model Predictive Control

For $k = 0, 1, \ldots$ solve the time k problem

$$\text{minimize } J_k = \sum_{l=k}^{k+N-1} f(\tilde{x}_l, \tilde{u}_l) \tag{5}$$

$$\begin{aligned} \text{subject to } & \tilde{x}_{l+1} = F(\tilde{x}_l, \tilde{u}_l),\ l = k, \ldots, k+N-1 \\ & \tilde{u}_l \in U,\ l = k, \ldots, k+N-1 \\ & \tilde{x}_l \in X,\ l = k+1, \ldots, k+N-1 \\ & \tilde{x}_{k+N} = 0 \end{aligned}$$

where $\tilde{x}_k = x_k$ is given. Denote the solution $\tilde{x}^\star_{k+1}, \ldots, \tilde{x}^\star_{k+N}, \tilde{u}^\star_k, \ldots, \tilde{u}^\star_{k+N-1}$ and let $u_k = \tilde{u}^\star_k$. The state evolves as

$$x_{k+1} = f(x_k, u_k), \quad k = 0, 1, \ldots$$

with x_0 given.

Example

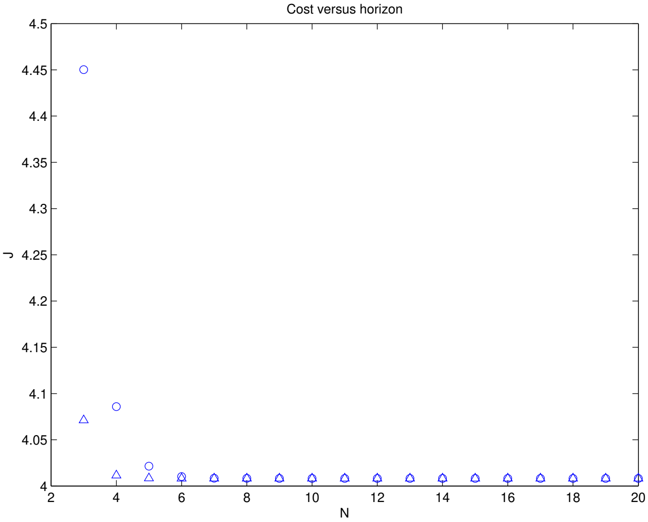
Dynamics: $F(x, u) = Ax + Bu$, where $A \in \mathbf{R}^{3\times 3}$ and $B \in \mathbf{R}^{3\times 2}$ random

Incremental cost: $f(x, u) = x^T x + u^T u$

Constraints: $-\mathbf{1} \leq x_k \leq \mathbf{1}$ and $-0.5 \times \mathbf{1} \leq u_k \leq 0.5 \times \mathbf{1}$

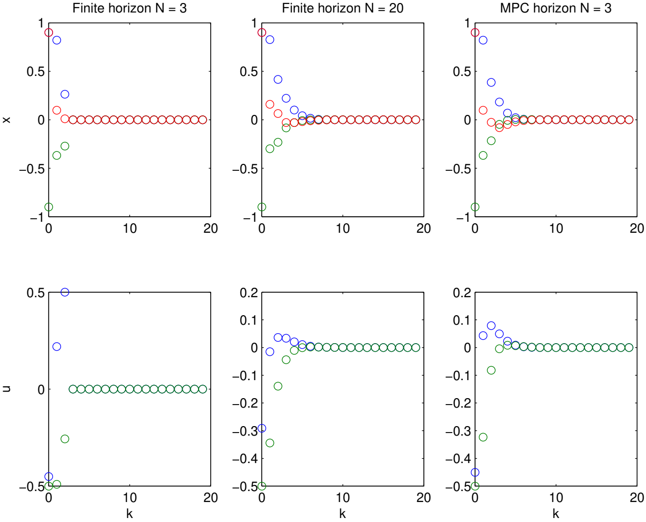
Initial value: $x_0 = \begin{bmatrix} 0.9 & -0.9 & 0.9 \end{bmatrix}^T$.

Optimal Cost



Finite time horizon approximation—circles, MPC—triangles

Trajectories



Multi-Parametric Programming

$$
\begin{aligned}
f^\star(\theta) \quad &= \min_z f(z,\theta) \\
&\text{subj. to } g(z,\theta) \le 0
\end{aligned}
$$

where $z \in \mathbf{R}^q$, $\theta \in \Theta \subseteq \mathbf{R}^r$ is a vector of parameters, and where $f : \mathbf{R}^q \times \mathbf{R}^r \to \mathbf{R}$ and $g : \mathbf{R}^q \times \mathbf{R}^r \to \mathbf{R}^s$.

Want to solve for all values of θ

Optimal value of z will depend on θ.

KKT Conditions

For convex (for all fixed values of θ) multi-parametric programs KKT conditions are necessary and sufficient conditions for optimality. i.e. there exist $z(\theta) \in \mathbf{R}^q$ and $\lambda(\theta) \in \mathbf{R}^s$ such that

$$\nabla_z f(z(\theta),\theta) + \lambda^T(\theta)\nabla_z g(z(\theta),\theta) = 0 \tag{6}$$

$$g(z(\theta),\theta) \leq 0 \tag{7}$$

$$\lambda(\theta) \geq 0 \tag{8}$$

$$\lambda_i(\theta) g_i(\theta) = 0, \quad i \in \mathcal{I} \tag{9}$$

where sub-script i denotes the i:th component of a vector, and where $\mathcal{I} = \{1, \ldots, s\}$.

Multi-Parametric Quadratic Program

Let $f = \frac{1}{2}z^T H z$, and $g = Gz - w - S\theta$, where $H \in \mathbf{R}^{q \times q}$ is a positive definite symmetric matrix, $G \in \mathbf{R}^{s \times q}$ and $S \in \mathbf{R}^{s \times r}$.

KKT conditions:

$$Hz + G^T \lambda = 0 \tag{10}$$

$$Gz \leq w + S\theta \tag{11}$$

$$\lambda \geq 0 \tag{12}$$

$$\lambda_i (G_i z - w_i - S_i \theta) = 0, \quad i = 1, \ldots, s \tag{13}$$

where sub-script i denotes the i:th row of a vector or matrix.

Features of Optimal Solution

- Optimizer function $z^\star(\theta)$ piecewise affine over polyhedral subsets of Θ
- Value function $f^\star(\theta)$ piecewise quadratic over polyhedral subsets of Θ

Explicit MPC

for the case when f is quadratic, F is linear, and X and U are described by linear constraints.

- Fedback is piecewise affine over polyhedral partitioning of the state-space
- Feedback can be computed off-line
- On-line it is only needed to compute which polyhedral region x_k is in
- Stability of MPC self-study