Optimal Control, Lecture 2: Dynamic Programming

Anders Hansson

Division of Automatic Control
Linkoping University

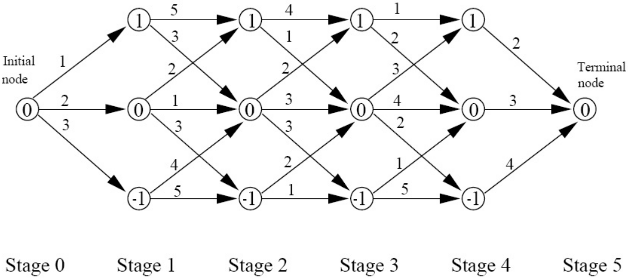
Contents

Multistage Decision Problem

$$\begin{array}{ll} \text{minimize} & \phi(x_N) + \sum_{k=0}^{N-1} f_k(x_k, u_k) \\ \text{subject to} & x_{k+1} = F_k(x_k, u_k) \\ & x_0 \text{ given } , x_k \in X_k \\ & u_k \in U(k, x_k) \end{array} \qquad (1)$$

k does not have to be time, but just enumeration of stages

Shortest Path Problem revisited



Find the shortest path from the initial node at stage 0 to the terminal node at stage 5.

and casted as Multistage Decision Problem

On white board

Principle of Optimality

If $\{u_k^*\}_{k=0}^{N-1}$ is optimal for (1), then $\{u_k^*\}_{k=n}^{N-1}$ is optimal for a problem on form (1) but with intial value (n, x_n^*) instead of $(0, x_0)$.

Optimal Cost-to-Go Function

$$
\begin{array}{ll}
J_l^*(x) = \text{min} & \phi(x_N) + \sum_{k=l}^{N-1} f_k(x_k, u_k) \\
\text{subject to} & x_{k+1} = F_k(x_k, u_k) \\
 & x_l = x, x_k \in X_k \\
 & u_k \in U_k(x_k)
\end{array} \qquad (2)
$$

for $l = 0, 1, \ldots, N-1$ with $J_N^*(x) = \phi(x)$

Notice optimal value of (1) is $J_0^*(x_0)$

Dynamic Programming

Suppose there exist finite solution to the backward Dynamic Programming recursion

$$V_N(x) = \begin{cases} \phi(x), & x \in X_N \\ \infty, & x \notin X_N \end{cases}$$

$$V_k(x) = \min_{u \in U_k(x), F_k(x,u) \in X_{k+1}} \{ f_k(x,u) + V_{k+1}(F_k(x,u)) \} \quad (3)$$

$k = N-1, N-2, \ldots, 0$ Then there exists an optimal solution to (1) and

- (a) $J_k^*(x) = V_k(x)$ for all $k = 0, 1, \ldots, N,\ x \in X_n$
- (b) The optimal feedback control in each stage is the minimizing argument in (3)

Proof, see Section 8.1.2.

Short Course on Gradients

$f(x + \delta x) = f(x) + (\nabla f)^T \delta x +$ higher order terms

For $f(x) = c^T x$ we get $\nabla f = c$

For $f(x) = x^T M x$ where M symmetric

$f(x + \delta x) = (x + \delta x)^T M (x + \delta x) =$
$x^T M x + x^T M \delta x + \delta x^T M x + \delta x^T M \delta x = x^T M x + 2(x^T M) \delta x +$
higher order term

Hence $\nabla f = 2Mx$

Infinite Time Horizon Oprimization

$$
\begin{aligned}
J^*(x_0) = \text{min} \quad & \textstyle\sum_{k=0}^{\infty} \gamma^k f(x_k, u_k) \\
\text{subject to} \quad & x_{k+1} = F(x_k, u_k) \\
& x_0 \text{ given} \\
& u_k \in U(x_k)
\end{aligned}
\tag{4}
$$

with $0 < \gamma \leq 1$ discount factor.

Bellman Equation

Assume $0 \in U(0)$, $f(0,0) = 0$, $F(0,0) = 0$ and that f is strictly positive definite. If there exists a strictly positive definite V such that the Bellman equation

$$V(x) = \min_{u \in U(x)} \{f(x,u) + \gamma V(F(x,u))\}$$

holds, then

- (a) $J^*(x) = V(x)$
- (b) The minimizing argument in the Bellman equation is an optimal feedback for (4) that results in a globally convergent closed loop system if γ is sufficiently close to one.