Optimal Control, Lecture 10: Generalizations of the PMP

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Contents

Optimal Control Problem

$$\begin{array}{ll} \text{minimize} & \phi\left(x(T)\right) + \int_0^T f\left(x(t), u(t)\right) dt \\ \text{subject to} & \dot{x}(t) = F\left(x(t), u(t)\right) \\ & x(0) \in S_0, \quad x(T) \in S_T \\ & u(t) \in U \subset \mathbf{R}^m \\ & T \geq 0 \end{array} \qquad (1)$$

with variables x, u, and T, where $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$, $F : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ and $\phi : \mathbf{R}^n \to \mathbf{R}$ are continuously differentiable. The sets S_0 and S_T are subsets of $\mathbf{R}^n$ and *manifolds*. We assume

$$S_0 = \left\{x \in \mathbf{R}^n : G_0(x) = 0\right\},$$

where $G_0 : \mathbf{R}^n \to \mathbf{R}^p$ with $p \leq n$ is differentiable with a full rank Jacobian for x in a neighborhood of the optimal solution point on S. We assume a similar description of S_T with a function G_T.

Hamiltonian

Define the Hamiltonian $H : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{n+1} \to \mathbf{R}$ as

$$H(x, u, \tilde{\lambda}) = \lambda_0 f(x, u) + \lambda^T F(x, u),$$

where $\tilde{\lambda} = (\lambda_0, \lambda)$.

PMP General Case

Assume that $(x^\star, u^\star, T^\star)$ are optimal for the optimal control problem above. Then there exist a nonzero adjoint function $\tilde{\lambda} : [0, T] \to \mathbf{R}^{n+1}$ such that

(i) $\dot{\lambda}(t) = -\frac{\partial H(x^\star(t), u^\star(t), \tilde{\lambda}(t))}{\partial x}$, $\lambda_0 = c \geq 0$, where $c \in \mathbf{R}$ is a constant

(ii) $H(x^\star(t), u^\star(t), \tilde{\lambda}(t)) = \min_{v \in U} H(x^\star(t), v, \tilde{\lambda}(t)) = 0, \ \forall t \in [0, T^\star]$

(iii) $\lambda(0) \perp S_0$

(iv) $\lambda(T) - \frac{\partial \phi(x^\star(T^\star))}{\partial x} \perp S_T$

Comments

Above we have used the notation $\lambda_0 \perp S_0$ to mean that $\lambda(0)^T v = 0$ for all v such that $\frac{\partial G_0(x(0))}{\partial x^T} v = 0$, where G_0 is the function defining the manifold S_0.

In case the final time T is not optimized, then condition (ii) is replaced with that the Hamiltonian is constant and not necessarily zero along the optimal solution.

For many problems it turns out that $\lambda_0 > 0$, and then since H is homogeneous in $\tilde{\lambda}$ there is no loss in generality to take $\lambda_0 = 1$.

Solution Procedure

1. Define Hamiltonian $H(x, u, \tilde{\lambda}) = \lambda_0 f(x, u) + \lambda^T F(x, u)$.
2. Let
$$u^\star = \mu(x, \tilde{\lambda}) = \underset{u}{\operatorname{argmin}} \, H(x, u, \tilde{\lambda})$$
3. Substitute $u^\star$ into the dynamical equations for x and $\tilde{\lambda}$, *i.e.*,

$$\begin{aligned} \dot{x} &= F(x, \mu(x, \tilde{\lambda})), \quad x(0) \in S_0, \; x(T) \in S_T \\ \dot{\lambda} &= -\frac{\partial H(x, \mu(x, \tilde{\lambda}), \tilde{\lambda})}{\partial x}, \quad \lambda(0) \perp S_0, \; \lambda(T) - \frac{\partial \phi(x(T))}{\partial x} \perp S_T \end{aligned} \tag{2}$$

Comments

- The two point boundary problem in (2) is in general not easy to solve
- Also use $H(x^\star(t), u^\star(t), \tilde{\lambda}(t)) = \min_{v \in U} H(x^\star(t), v, \tilde{\lambda}(t)) = 0, \ \forall t \in [0, T^\star]$ if possible.
- To carry out the parametric optimization of H can be very difficult, especially when u is constrained by the set U.
- Remember that the PMP are only necessary conditions for optimality.
- Hence they may not provide enough information to uniquely determine the optimal control.
- Moreover, they do not guarantee optimality, but only stationarity.
- Further investigations are necessary to prove that a candidate solution obtained from the PMP is indeed optimal.

Examples

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