Optimal Control - Explicit MPC (Exercise)

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In this exercise we will focus on the explicit solution to the *model predictive control* (MPC) problem:

$$
\begin{aligned}
\underset{u(\cdot)}{\text{minimize}} \quad & \sum_{l=k}^{k+N-1} \left(x_l^T Q x_l + u_l^T R u_l \right) \\
\text{subject to} \quad & x_{l+1} = A x_l + B u_l, \\
& u_l \in \mathcal{U}, \\
& x_l \in \mathcal{X},
\end{aligned} \tag{1}
$$

for which the basics are given in [1]. The MPC formulation (1) can be reformulated as a *multi-parametric quadratic program* (mpQP)

$$
\begin{aligned}
\underset{z}{\text{minimize}} \quad & \frac{1}{2} z^T H z \\
\text{subject to} \quad & Gz \leq w + S\theta.
\end{aligned} \tag{2}
$$

Now, consider finding the analytical solution to the explicit MPC problem. Let the system be described by the dynamics

$$
x_{k+1} = x_k + u_k, \tag{3}
$$

where $\dim(x_k) = 1$ and $\dim(u_k) = 1$.

a) Formulate the MPC problem (1) for the dynamical system (3) as an mpQP (as described in Section 8.8 [1]) for the prediction horizon $N = 2$ with the constraint $\mathcal{U} = \{u \in \mathbb{R} : |u| \leq 1\}$ on the control signal, i.e., determine H, G, w and S as functions of the variables Q and R, respectively.

   Hint: Use some symbolic software, e.g., MAPLE or MATHEMATICA.

b) Find the solution z^* to the problem derived in a) for the case when $\theta = 0$.

c) Determine the region $A\theta \leq b$ where the solution above is valid, that is, determine A and b (see Section 5.6 [1]), where

$$
A = \begin{bmatrix} GH^{-1}G_{\mathcal{A}}^T \left(G_{\mathcal{A}} H^{-1} G_{\mathcal{A}}^T \right)^{-1} S_{\mathcal{A}} - S \\ \left(G_{\mathcal{A}} H^{-1} G_{\mathcal{A}}^T \right)^{-1} S_{\mathcal{A}} \end{bmatrix} \tag{4a}
$$

$$
b = \begin{bmatrix} w - GH^{-1}G_{\mathcal{A}}^T \left(G_{\mathcal{A}} H^{-1} G_{\mathcal{A}}^T \right)^{-1} w_{\mathcal{A}} \\ - \left(G_{\mathcal{A}} H^{-1} G_{\mathcal{A}}^T \right)^{-1} w_{\mathcal{A}} \end{bmatrix} \tag{4b}
$$

d) Show that the optimal control is given by

$$u^* = \begin{cases} 1, & x_k \leq -\frac{Q+R}{Q} \\ -\frac{Q}{Q+R} x_k, & -\frac{Q+R}{Q} \leq x_k \leq \frac{Q+R}{Q} \\ -1, & x_k \geq \frac{Q+R}{Q} \end{cases}$$

e) Verify the solution using the MPT toolbox in MATLAB for $(Q = 2, R = 1)$ and $(Q = 1, R = 2)$.

f) We see in d) that the optimal control only depends on R/Q. Explain how this can be seen in a more direct way in (1).

References

[1] Hansson A. and Andersen M. Optimization for learning and control. John Wiley & Sons, 2023.