TSRT08: Optimal Control
Solutions

20240827

1. (a) The Hamiltonian is given by

$$\begin{aligned} H(t,x,u,\lambda) &\triangleq f(t,x,u) + \lambda^T F(t,x,u) \\ &= (x(t) - \cos t)^2 + u^2(t) + \lambda u \end{aligned}$$

Pointwise minimization yields

$$\tilde{\mu}(t,x,\lambda) \triangleq \arg\min_u \quad H(t,x,u,\lambda) = -\frac{1}{2}\lambda,$$

and the optimal control can be written as

$$u^*(t) \triangleq \tilde{\mu}(t,x(t),\lambda(t)) = -\frac{1}{2}\lambda(t)$$

The adjoint equation is given by

$$\dot{\lambda}(t) = -2(x(t) - \cos t),$$

since $\dot{x}(t) = u(t)$, therefore $\dot{x} = -\frac{1}{2}\lambda$, then

$$\ddot{x}(t) = \dot{u}(t) = -\frac{1}{2}\dot{\lambda} = x(t) - \cos t$$

which is a linear ordinary differential equation with $x(0) = 0$ and $\dot{x}(t_f) = u(t_f) = -\frac{1}{2}\lambda(t_f) = 0$.

(b) The Hamiltonian is given by

$$\begin{aligned} H(t,x,u,\lambda) &\triangleq f(t,x,u) + \lambda^T F(t,x,u) \\ &= \frac{1}{2}u^2(t) + \lambda u \end{aligned}$$

Pointwise minimization yields

$$\tilde{\mu}(t,x,\lambda) \triangleq \arg\min_u \quad H(t,x,u,\lambda) = -\lambda,$$

and the optimal control can be written as

$$u^*(t) \triangleq \tilde{\mu}(t,x(t),\lambda(t)) = -\lambda(t)$$

The adjoint equation is given by

$$\dot{\lambda}(t) = 0,$$

which means that λ is constant. The boundary condition is given by

$$\lambda\left(t_f\right)=\frac{\partial \phi}{\partial x}\left(x\left(t_f\right)\right)=\gamma x\left(t_f\right) \Rightarrow \lambda(t)=\gamma x\left(t_f\right)$$

The final state $x\left(t_f\right)$ is not given, however it can be calculated as a function of $x(t)$ assuming the optimal control signal $u^*(t)=-\lambda(t)=-\gamma x\left(t_f\right)$ is applied from t to t_f. This will also give us a feedback solution.

$$x\left(t_f\right)=x(t)+\int_t^{t_f} \dot{x}(t) d t=x(t)-\int_t^{t_f} \gamma x\left(t_f\right) d t=x(t)-\gamma x\left(t_f\right)\left(t_f-t\right).$$

Thus,

$$x\left(t_f\right)=\frac{1}{1+\gamma\left(t_f-t\right)} x(t),$$

and the optimal control is found by

$$u^*(t)=-\lambda(t)=-\gamma x\left(t_f\right)=-\frac{1}{\gamma^{-1}+\left(t_f-t\right)} x(t).$$

2. From hint, we define $z(t)=\int_{-a}^t \sqrt{1+\dot{x}^2(s)} d s$, then we have

$$\dot{z}(t)=\sqrt{1+\dot{x}^2} \tag{2.1}$$

let $\dot{x}=u$, then an equivalent problem is

$$\begin{aligned}
\underset{x(.)}{\operatorname{minimize}} & -\int_{-a}^a x(t) d t \\
\text{subject to } & \dot{x}(t)=u(t), \\
& x(-a)=0, \\
& x(a)=0, \\
& \dot{z}(t)=\sqrt{1+u(t)^2} \\
& z(-a)=0 \\
& z(a)=l
\end{aligned} \tag{2.2}$$

Thus, the Hamiltonian is

$$H=-x+\lambda_1 u+\lambda_2 \sqrt{1+u^2} \tag{2.3}$$

and the following equations must hold

$$\dot{\lambda}_1=-\frac{\partial H}{\partial x}=1 \tag{2.4a}$$

$$\dot{\lambda}_2=-\frac{\partial H}{\partial z}=0 \tag{2.4b}$$

$$0=\frac{\partial H}{\partial u}=\lambda_1+\lambda_2 u\left(1+u^2\right)^{-1/2} \tag{2.4c}$$

Equations (2.4a) and (2.4b) gives $\lambda_1=t+c_1$ and $\lambda_2=c_2$ where c_1 and c_2 are real constants. These solutions inserted in (2.4c) gives

$$\left(t+c_1\right)\left(1+u^2\right)^{1/2}+u c_2=0 \tag{2.5}$$

and after some rearrangement and taking the square we have

$$u^2 \left(c_2^2 - (t + c_1)^2\right) = (t + c_1)^2 \tag{2.6}$$

Note that $c_2^2 - (t + c_1)^2 \leq 0$ leads to the unrealistic requirement $t = c_1$. Hence, we can assume that $c_2^2 - (t + c_1)^2 > 0$ and rewrite (2.6) as

$$\dot{x} = u = \pm \frac{t + c_1}{\sqrt{c_2^2 - (t + c_1)^2}} \tag{2.7}$$

The solution is

$$x(t) = \pm \sqrt{c_2^2 - (t + c_1)^2} + c_3 \tag{2.8}$$

where c_3 is constant, and this can be rewritten as the equation of a circle with the radius c_2 and the center in $(t, x(t)) = (c_1, c_3)$, i.e.,

$$(x(t) - c_3)^2 + (t + c_1)^2 = c_2^2 \tag{2.9}$$

The requirements $x(-a) = 0$ and $x(a) = 0$ gives

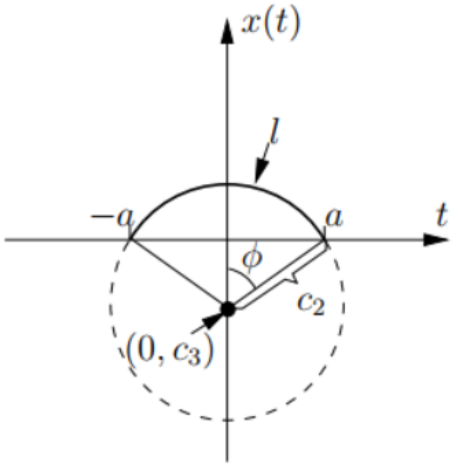
$$\begin{aligned} c_3^2 + (a - c_1)^2 &= c_2^2 \\ c_3^2 + (a + c_1)^2 &= c_2^2 \end{aligned} \tag{2.10}$$

respectively. Thus, $c_1 = 0$ and $c_3^2 + a^2 = c_2^2$. We have three different cases

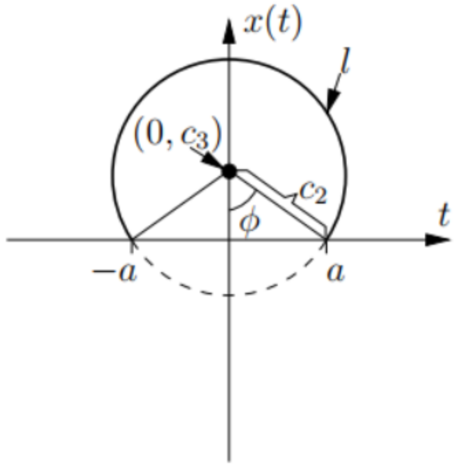
- $l < 2a$: No solution since the distance between $(-a, 0)$ and $(a, 0)$ is longer than the length of the curve l.
- $2a \leq l \leq \pi a$: Let ϕ define the angle from the $x(t)$-axis to the vector from the circle center to the point $(a, 0)$, i.e, $a = c_2 \sin \phi$, see Figure 1 (a). Furthermore, the length of the circle segment curve is $l = 2\phi c_2$. Thus

  $$\sin \frac{l}{2c_2} = \frac{a}{c_2}$$

  which can be used to determine c2.
- $\pi a < l$: as the previous case, but the length of the circle segment curve is now $l = 2(\pi - \phi)c_2$, see Figure 1(b).



(a) $2a \leq l \leq \pi a$



(b) $\pi a < l$

Figure 1: Optimal curve enclosing the largest area in the upper half plane for two different cases.

3. (a) We define the error signal as

$$\varepsilon(u) = y - r$$

we are now investigate in finding u such that the error signal is zero for the given reference signal r. The following root finding algorithm can be used

$$u_{k+1} = u_k - t\varepsilon(u_k)$$

where the subindex k refers to iteration index and not to stage index or time. Notice that the computations involved in the root-finding algorithm only require us to be able to evaluate the error ε for a known input u. This can be done with simulations or with experiments on a real dynamical system.

(b) Assume that H is Lipschitz continuous with Lipschitz constant β and that H is strongly convex with parameter α. Then the root finding algorithm converges if $t \in (0, \frac{2}{\alpha+\beta})$, see Section 11.6 in the course book.

4. We have the optimization problem on standard form with $N = 4$, $\phi(x) = 0$,$f(k, x, u) = x^2 + u^2$, and $F(k, x, u) = x+u$. Note that $0 \leq x_k+u_k \leq 5 \iff 0 \leq x_{k+1} \leq 5$. So we know that $0 \leq x_k \leq 5$ for $k = 0, 1, 2, 3$ if $u_k \in U(k, x)$.

Stage $k = N = 4$: $V(4, x) = 0$

Stage $k = 3$:

$$V(3, x) = \min_{u \in U(3,x)} \{f(3, x, u) + V(4, F(3, x, u))\} = \min_{u \in U(3,x)} \{x^2 + u^2\} = x^2$$

Since $u^* = 0 \in U(3, x)$ for all x satisfying $0 \leq x \leq 5$.

Stage $k = 2$:

$$\begin{aligned} V(2, x) &= \min_{u \in U(2,x)} \{f(2, x, u) + V(3, F(2, x, u))\} \\ &= \min_{u \in U(2,x)} \left\{x^2 + u^2 + (x + u)^2\right\} = \frac{3}{2}x^2 + \min_{-x \leq u \leq 5-x} \left\{2(\frac{1}{2}x + u)^2\right\} \end{aligned}$$

x	$V(2, x)$	u^*
0	$\min_{0 \leq u \leq 5}\{2u^2\} = 0$	0
1	$\frac{3}{2} + \min_{-1 \leq u \leq 4}\{2(\frac{1}{2} + u)^2\} = 2$	-1, 0
2	$6 + \min_{-2 \leq u \leq 3}\{2(1 + u)^2\} = 6$	-1
3	$\frac{27}{2} + \min_{-3 \leq u \leq 2}\{2(\frac{3}{2} + u)^2\} = 14$	-1, -2
4	$24 + \min_{-4 \leq u \leq 1}\{2(2 + u)^2\} = 24$	-2
5	$\frac{75}{2} + \min_{-5 \leq u \leq 0}\{2(\frac{5}{2} + u)^2\} = 38$	$-3, -2$

Stage $k = 1$:

$$V(1, x) = \min_{u \in U(1,x)} \{f(1, x, u) + V(2, F(1, x, u))\} = x^2 + \min_{-x \leq u \leq 5-x} \left\{u^2 + V(2, x + u\right\}$$

x	$V(1, x)$	u^*
0	$0 + \min_{0 \leq u \leq 5}\{u^2 + V(2, 0 + u)\} = 0$	0
1	$1 + \min_{-1 \leq u \leq 4}\{u^2 + V(2, 1 + u)\} = 2$	-1
2	$4 + \min_{-2 \leq u \leq 3}\{u^2 + V(2, 2 + u)\} = 7$	-1
3	$9 + \min_{-3 \leq u \leq 2}\{u^2 + V(2, 3 + u)\} = 15$	-2
4	$16 + \min_{-4 \leq u \leq 1}\{u^2 + V(2, 4 + u)\} = 26$	-2
5	$25 + \min_{-5 \leq u \leq 0}\{u^2 + V(2, 5 + u)\} = 40$	-3

since the expression inside the brackets $u^2 + V(2, x + u)$ are for different x and u,

x	$u=-5$	-4	-3	-2	-1	0	1	2	3	4	5
0						$\mathbf{0+0}$	$1+2$	$4+6$	$9+14$	$16+24$	$25+38$
1					$\mathbf{1+0}$	$0+2$	$1+6$	$4+14$	$9+24$	$16+38$	
2				$4+0$	$\mathbf{1+2}$	$0+6$	$1+14$	$4+24$	$9+38$		
3			$9+0$	$\mathbf{4+2}$	$1+6$	$0+14$	$1+24$	$4+38$			
4		$16+0$	$9+2$	$\mathbf{4+6}$	$1+14$	$0+24$	$1+38$				
5	$25+0$	$16+2$	$\mathbf{9+6}$	$4+14$	$1+24$	$0+38$					

Stage $k=0$:

$$V(0,x) = \min_{u \in U(0,x)} \{f(0,x,u) + V(1, F(1,x,u))\}$$

note that, $x_0 = 5$, therefore we have

$$V(0,x) = 25 + \min_{-5 \le u \le 0} \left\{u^2 + V(1, 5+u)\right\} = 41$$

since the expression inside the brackets $u^2 + V(1, 5+u)$ are for $x=5$ and different u,

x	$u=-5$	-4	-3	-2	-1	0
0	$25+0$	$16+2$	$\mathbf{9+7}$	$4+15$	$1+26$	$0+40$

To summarize, the system evolves as follows

k	x_k	u_k	V_k
0	5	-3	41
1	2	-1	7
2	1	-1 or 0	2
3	0 or 1	0	0 or 1