TSRT08: Optimal Control
Solutions

20240313

1. (a) The Hamiltonian is given by

$$\begin{aligned} H(t,x,u,\lambda) &\triangleq f(t,x,u) + \lambda^T F(t,x,u) \\ &= x + u^2 + \lambda x + \lambda u + \lambda \end{aligned}$$

Pointwise minimization yields

$$\tilde{\mu}(t,x,\lambda) \triangleq \arg\min_u \quad H(t,x,u,\lambda) = -\frac{1}{2}\lambda,$$

and the optimal control can be written as

$$u^*(t) \triangleq \tilde{\mu}(t,x(t),\lambda(t)) = -\frac{1}{2}\lambda(t)$$

The adjoint equation is given by

$$\dot{\lambda}(t) = -\lambda(t) - 1,$$

which is a first order linear ODE and can thus be solved easily. The standard integrating factor method yields

$$\lambda(t) = e^{C-t} - 1,$$

for some constant C. The boundary condition is given by

$$\lambda(T) = \frac{\partial \phi}{\partial x}(T, x(T)) \quad \Longleftrightarrow \lambda(T) = 0,$$

Thus,

$$\lambda(t) = e^{T-t} - 1,$$

and the optimal control is found by

$$u^*(t) = -\frac{1}{2}\lambda(t) = \frac{1 - e^{T-t}}{2}.$$

(b) i. Since

$$J = \int_0^1 \dot{y} dt = y(1) - y(0) = 1,$$

it holds that all $y(\cdot) \in C^1[0,1]$ such that $y(0) = 0$ and $y(1) = 1$ are minimal.

ii. Since

$$J = \int_0^1 y\dot{y} dt = \frac{1}{2}\int_0^1 \frac{d}{dt}(y^2) dt = \frac{1}{2}(y(1) - y(0))^2 = \frac{1}{2},$$

it holds that all $y(\cdot) \in C^1[0,1]$ such that $y(0) = 0$ and $y(1) = 1$ are minimal.

2. The problem to be solved is

$$\begin{array}{ll}
\underset{u(\cdot)}{\text{minimize}} & \int_0^T u(t)dt \\
\text{subject to} & \dot{x}_1(t) = x_2(t), \\
& \dot{x}_2(t) = \frac{cu(t)}{x_3(t)} - g\left(1 - kx_1(t)\right), \\
& \dot{x}_3(t) = -u(t), \\
& x_1(0) = h, x_2(0) = \nu, x_3(0) = m, \\
& x_1\left(T\right) = 0, x_2\left(T\right) = 0, \\
& 0 \leq u(t) \leq M.
\end{array}$$

The Hamiltonian is given by

$$\begin{aligned}
H(t,x,u,\lambda) &\triangleq f(t,x,u) + \lambda^T F(t,x,u) \\
&= u + \lambda_1 x_2 + \lambda_2\left(\frac{cu}{x_3} - g\left(1 - kx_1\right)\right) - \lambda_3 u \\
&= \lambda_1 x_2 - \lambda_2 g\left(1 - kx_1\right) + \left(1 + \frac{c\lambda_2}{x_3} - \lambda_3\right) u.
\end{aligned}$$

Pointwise minimization yields

$$\begin{aligned}
\tilde{\mu}(t,x,\lambda) &\triangleq \underset{u\in[0,M]}{\arg\min} \quad H(t,x,u,\lambda) \\
&= \underset{0\leq u\leq M}{\arg\min} \quad \underbrace{\left(1 + \frac{c\lambda_2}{x_3} - \lambda_3\right)}_{\triangleq \sigma} u \\
&= \begin{cases} M, & \sigma < 0 \\ 0, & \sigma > 0 \\ \tilde{u} & \sigma = 0 \end{cases}
\end{aligned}$$

where $\tilde{u} \in [0,M]$ is arbitrary. Thus, the optimal control is expressed by

$$u^*(t) \triangleq \tilde{\mu}(t, x(t), \lambda(t)) = \left\{ \begin{array}{ll} M, & \sigma(t) < 0 \\ 0, & \sigma(t) > 0 \\ \tilde{u}, & \sigma(t) = 0 \end{array} \right. ,$$

where the switching function is given by

$$\sigma(t) \triangleq 1 + \frac{c\lambda_2(t)}{x_3(t)} - \lambda_3(t)$$

The adjoint equations are

$$\begin{aligned}
\dot{\lambda}_1(t) &= -\frac{\partial H}{\partial x_1}\left(x(t), u^*(t), \lambda(t)\right) = -gk\lambda_2(t), \\
\dot{\lambda}_2(t) &= -\frac{\partial H}{\partial x_2}\left(x(t), u^*(t), \lambda(t)\right) = -\lambda_1(t) \\
\dot{\lambda}_3(t) &= -\frac{\partial H}{\partial x_3}\left(x(t), u^*(t), \lambda(t)\right) = \frac{c\lambda_2(t)u^*(t)}{x_3^2(t)}
\end{aligned}$$

The boundary conditions are given by

$$\lambda\left(T\right) - \frac{\partial \phi}{\partial x}\left(T, x\left(T\right)\right) \perp S_f\left(T\right) \Longleftrightarrow \begin{pmatrix} \lambda_1\left(T\right) \\ \lambda_2\left(T\right) \\ \lambda_3\left(T\right) \end{pmatrix} = \nu_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \nu_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

which says that $\lambda_3\left(T\right) = 0$, while $\lambda_1\left(T\right)$ and $\lambda_2\left(T\right)$ are free. This, yields no particular information about the solution to the adjoint equations.

Now, let us try to determine the number of switches. Using the state dynamics and the adjoint equations, it follows that

$$\begin{aligned}\dot{\sigma}(t) &= \frac{c\dot{\lambda}_2(t)x_3(t) - c\lambda_2(t)\dot{x}_3(t)}{x_3^2(t)} - \dot{\lambda}_3(t) \\ &= \frac{-c\lambda_1(t)x_3(t) + c\lambda_2(t)u^*(t)}{x_3^2(t)} - \frac{c\lambda_2(t)u^*(t)}{x_3^2(t)} \\ &= -c\frac{\lambda_1(t)}{x_3(t)}.\end{aligned}$$

Since $c > 0$ and $x_3(t) \geq 0$, it holds that $\operatorname{sign}\dot{\sigma}(t) = -\operatorname{sign}\lambda_1(t)$ and we need to know which values $\lambda_1(t)$ can take. From the adjoint equations, we have

$$\ddot{\lambda}_1(t) = -gk\dot{\lambda}_2(t) = gk\lambda_1(t)$$

which has the solution

$$\lambda_1(t) = Ae^{\sqrt{gk}t} + Be^{-\sqrt{gk}t}$$

Now, if A and B have the same signs, $\lambda_1(t)$ will never reach zero. If A and B have opposite signs, $\lambda_1(t)$ has one isolated zero. This implies that $\dot{\sigma}(t)$ has at most one isolated zero. Thus, $\sigma(t)$ can only, at the most, pass zero two times and the optimal control is bang-bang. The only possible sequences are thus $\{M, 0, M\}, \{0, M\}$ and $\{M\}$ (since the spacecraft should be brought to rest, all sequences ending with a 0 are ruled out).

3. Define

$$x_k = \begin{cases} 0 & \text{if on activity } g_1 \text{ just before time } t_k \\ 1 & \text{if on activity } g_2 \text{ just before time } t_k \end{cases},$$

$$u_k = \begin{cases} 0 & \text{continue current activity,} \\ 1 & \text{switch between activities} \end{cases}$$

The state transition is

$$x_{k+1} = F(x_k, u_k) = (x_k + u_k) \quad \text{mode } 2$$

and the profit for stage k is

$$f(x_k, u_k) = \int_{t_k}^{t_{k+1}} g_{1+F(x_k,u_k)}(t)dt - u_k c$$

The dynamic programming is then

$$\begin{aligned}&V_N(x_N) = 0 \\ &V_n(x) = \max_u\{f(x, u) + V_{n+1}(x + u)\} \quad \text{mode } 2\end{aligned}$$

4. (a) we start from the Bellman equation

$$V(x) = \min_u \quad Q(x, u)$$

we have

$$\gamma V(F(x, \bar{u})) = \min_u \quad \gamma Q(F(x, \bar{u}), u)$$

By adding $f(x, \bar{u})$ to both sides of the above equation, the left hand side becomes $Q(x, \bar{u})$, and therefore we have

$$Q(x, \bar{u}) = f(x, \bar{u}) + \min_u \quad \gamma Q(F(x, \bar{u}), u)$$

(b) Let start with the assumption

$$Q_1(F(x,\bar{u}),u) \leq Q_2(F(x,\bar{u}),u)$$

minimizing both side of the equation with respect to u yields

$$\min_u \quad Q_1(F(x,\bar{u}),u) \leq \min_u \quad Q_2(F(x,\bar{u}),u)$$

hence by multiplying the both sides of the inequality by a factor γ and adding $f(x,\bar{u})$ to both sides, we get

$$f(x,\bar{u}) + \min_u \quad \gamma Q_1(F(x,\bar{u}),u) \leq f(x,\bar{u}) + \min_u \quad \gamma Q_2(F(x,\bar{u}),u)$$

also since $T_Q(Q)(x,\bar{u}) = f(x,\bar{u}) + \min_u \quad \gamma Q(F(x,\bar{u}),u)$, then we obtain

$$T_Q(Q_1)(x,\bar{u}) \leq T_Q(Q_2)(x,\bar{u})$$