EXAM IN OPTIMAL CONTROL (TSRT08)

ROOM: FE249, TER2

TIME: Wednesday, March 13, 2024, 08.00-12.00

COURSE: TSRT08 OPTIMAL CONTROL

CODE: TEN1

DEPARTMENT: ISY

NUMBER OF EXERCISES: 4

NUMBER OF PAGES (including cover page): 3

EXAMINER: Anders Hansson, 070-3004401

VISITS: 09.00 and 11.00 by Reza Jafari

COURSE ADMINISTRATOR: Ninna Stensgård 013-282225, ninna.stensgard@liu.se

APPROVED TOOLS: Collections of formulas and tables printed by a publishing house, calculator. No other books are allowed.

SOLUTIONS: Linked from the course home page after the exam.

The exam can be inspected and checked out 2024-04-04, Room 2A:573, B-building, entrance 25, A-corridor, to the right.

PRELIMINARY GRADING:	grade 3	15 points
	grade 4	23 points
	grade 5	30 points

All solutions should be well motivated. Writing should be neat and clean.

Good Luck!

1. (a) Solve the optimal control problem

$$\begin{aligned} \text{minimize} \quad & \int_0^T \left(x(t) + u^2(t) \right) dt \\ \text{subject to} \quad & \dot{x}(t) = x(t) + u(t) + 1, \\ & x(0) = 0. \end{aligned}$$

(6p)

(b) Find the extremal of the functionals

i. $\int_0^1 \dot{y}\, dt$

ii. $\int_0^1 y\dot{y}\, dt$

when $y(0) = 0$ and $y(1) = 1$. (4p)

2. A spacecraft approaching the face of the moon can be described by the following equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= \frac{cu(t)}{x_3(t)} - g(1 - kx_1(t)), \\ \dot{x}_3(t) &= -u(t), \end{aligned}$$

where $u(t) \in [0, M]$ for all t, and the initial conditions are given by

$$x(0) = (h, \nu, m)^T$$

with the positive constants c, g, k, M, h, ν and m. The state x_1 is the altitude above the surface, x_2 the velocity and x_3 the mass of the spacecraft. Calculate the structure of the fuel minimizing control that brings the craft to rest on the surface of the moon. The fuel consumption is

$$\int_0^T u(t)\, dt,$$

and the time T is to be optimized. Show that the optimal control law is bang-bang with at most two switches. (10p)

3. A decision maker must continually choose between two activites over a time interval $[0, T]$. Choosing activity i at time t, where $i = 1, 2$, earns a reward at a rate $g_i(t)$, and every switch between the two activity costs $c > 0$. Thus, for example, the reward for starting with activity 1, switching to 2 at time t_1, and switching back to 1 at time $t_2 > t_1$ earns total reward

$$\int_0^{t_1} g_1(t)\,dt + \int_{t_1}^{t_2} g_2(t)\,dt + \int_{t_2}^{T} g_1(t)\,dt - 2c.$$

We want to find a set of switching times that maximize the total reward. Assume that the function $g_1(t) - g_2(t)$ changes sign a finite number of times in the interval $[0, T]$. Formulate the problem as a optimization problem and write down the corresponding dynamic programming recursion.
Hint: Let $t_1 < t_2 < \ldots < t_{N-1}$ denote the times where $g_1(t) = g_2(t)$. It is never optimal to switch activity at any other times! We can therefore divide the problem into $N-1$ stages, where we want to determine for each stage whether or not to switch. (10p)

4. (a) Consider the Bellman equation

$$V(x) = \min_u Q(x, u)$$

where $Q(x, u) = f(x, u) + \gamma V(F(x, u))$ for some functions f and F. Show that the following equation holds

$$Q(x, \bar{u}) = f(x, \bar{u}) + \min_u \gamma Q(F(x, \bar{u}), u)$$

if the Bellman equation holds. (4p)

(b) Let T_Q be defined as

$$T_Q(Q)(x, \bar{u}) = f(x, \bar{u}) + \min_u \gamma Q(F(x, \bar{u}), u)$$

and show that

$$T_Q(Q_1)(x, \bar{u}) \leq T_Q(Q_2)(x, \bar{u})$$

for all $(x, \bar{u})$, if $Q_1(x, \bar{u}) \leq Q_2(x, \bar{u})$ for all $(x, \bar{u})$. (6p)