TSRT08: Optimal Control
Solutions

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1. (a) The optimal control problem can be stated as

$$\begin{aligned}
\text{minimize} \quad & \int_0^1 -y(t)dt \\
\text{subject to} \quad & \dot{x}(t) = V\cos u(t) \\
& \dot{y}(t) = V\sin u(t) \\
& x(0) = 0, \quad x(1) = 1 \\
& y(0) = 0, \quad y(1) = 0
\end{aligned}$$

The Hamiltonian is given by

$$H(t, x, u, \lambda) = -y + \lambda_x V\cos u + \lambda_y V\sin u$$

Pointwise minimization yields

$$0 = \frac{\partial H}{\partial u} = -\lambda_x V\sin u + \lambda_y V\cos u \quad \Rightarrow \quad \tan u = \frac{\lambda_y}{\lambda_x}$$

The adjoint equation is given by

$$\begin{aligned}
\dot{\lambda}_x(t) &= -\frac{\partial H}{\partial x} = 0 \\
\dot{\lambda}_y(t) &= -\frac{\partial H}{\partial y} = 1
\end{aligned} \quad \Rightarrow \quad \begin{aligned}
\lambda_x(t) &= a_1 \\
\lambda_y(t) &= a_2 + t
\end{aligned} \tag{1}$$

for some constants a_1 and a_2. We then get

$$\tan u(t) = \frac{\lambda_y(t)}{\lambda_x(t)} = \frac{a_2 + t}{a_1} = \underbrace{\frac{a_2}{a_1}}_{:=c_1} + \underbrace{\frac{1}{a_1}}_{:=c_2} t = c_1 + c_2 t$$

for some constants c_1 and c_2.

(b) We use PMP to find the open loop control.
The Hamiltonian is given by

$$H(t, x, u, \lambda) = \lambda_1 x_2 + \lambda_2 u$$

Pointwise minimization yields

$$\tilde{\mu}(t, x, \lambda) = \arg\min_{|u|<1} H(t, x, u, \lambda) = \begin{cases} -1, & \lambda_2 > 0 \\ 1, & \lambda_2 < 0 \\ \tilde{u}, & \lambda_2 = 0 \end{cases},$$

where $\tilde{u}$ is arbitrary in $[-1, 1]$. Thus the optimal control is expressed as

$$u^*(t) \triangleq \tilde{\mu}(t, x(t), \lambda(t)) = \begin{cases} -1, & \sigma(t) > 0 \\ 1, & \sigma(t) < 0 \\ \tilde{u}, & \sigma(t) = 0 \end{cases}$$

where we have defined the switching function as

$$\sigma(t) \triangleq \lambda_2(t)$$

The adjoint equation is given by

$$\begin{aligned} \dot{\lambda}_1(t) &= -\frac{\partial H}{\partial x_1} = 0, \\ \dot{\lambda}_2(t) &= -\frac{\partial H}{\partial x_2} = -\lambda_1(t) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \lambda_1(t) &= c_1 \\ \lambda_2(t) &= -c_1 t + c_2 \end{aligned} \tag{2}$$

and the boundary constraints are

$$\lambda(1) - \frac{\partial \phi}{\partial x}(1, x(1)) \perp S_f \quad \Longleftrightarrow \quad \begin{pmatrix} \lambda_1(1) \\ \lambda_2(1) \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \nu \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \nu \in \mathbb{R}$$

which implies that $\lambda_2(1) = 1$ and $\lambda_1(1)$ is free. Thus,

$$\sigma(1) = \lambda_2(1) = 1 > 0$$

which gives that $u^*(1) = -1$. Since $\dot{\sigma}(t) = \dot{\lambda}_2(t) = -c_1$ it follows that at most one switch will occur. We then get

$$u^*(t) = \begin{cases} 1, & 0 \leq t \leq t' \\ -1, & t' \leq t \leq 1 \end{cases}, \tag{3}$$

for some unknown switching time $t' \in [0, 1]$. Note, if no switch would occur, this can still be described with (3) using $t' = 0$.

To find a value of t' we need to take the constraint $x_1(1) = 0$ into consideration. Consequently, we have to find $x(t)$ when the control (3) is used. For $t \in [0, t']$ we have

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= 1 \\ x_1(0) &= 0 \\ x_2(0) &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1(t) &= \frac{t^2}{2} \\ x_2(t) &= t \end{aligned}$$

and for $t \in [t', 1]$ we then get

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -1 \\ x_1(t') &= \frac{t'^2}{2} \\ x_2(t') &= t' \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1(t) &= -\frac{t^2}{2} + 2t't - t'^2 \\ x_2(t) &= -t + 2t' \end{aligned}$$

We now have

$$x_1(1) = -\frac{1}{2} + 2t' - t'^2 = 0 \quad \Rightarrow \quad t' = 1 \pm \sqrt{1 - \frac{1}{2}} = 1 - \frac{1}{\sqrt{2}}$$

where one solution has been excluded since $t' \in [0, 1]$. We then get the open loop control

$$u^*(t) = \begin{cases} 1, & 0 \leq t \leq 1 - \frac{1}{\sqrt{2}} \\ -1, & 1 - \frac{1}{\sqrt{2}} \leq t \leq 1. \end{cases}$$

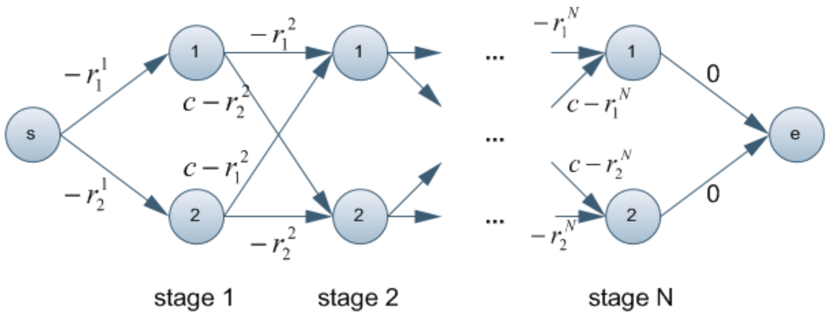

Figure 1: The shortest path problem in exercise 2.

2. (a) The shortest path from s to e maximizes the total profit over N days, see figure 1. The corresponding dynamic programming algorithm is

$$
\begin{aligned}
J(N,i) &= 0 \\
J(k,i) &= \min\{\underbrace{-r_i^{k+1} + J(k+1,i)}_{\triangleq q_i^{k+1},\ stay},\ \underbrace{c - r_{\bar{i}}^{k+1} + J(k+1,\bar{i})}_{\triangleq q_{\bar{i}}^{k+1},\ switch}\} \\
J(0,s) &= \min\{-r_1^1 + J(1,1),\ \ -r_2^1 + J(1,2)\}
\end{aligned}
$$

(b) Consider the difference $Q_i^k = q_i^k - q_{\bar{i}}^k$. If $Q_i^k \leq 0$ it is optimal to stay in i, and if $Q_i^k \geq 0$ it is optimal to switch to $\bar{i}$.

$$
\begin{aligned}
Q_i^k &= q_i^k - q_{\bar{i}}^k \\
&= -r_i^k + J(k,i) - c + r_{\bar{i}}^k - J(k,\bar{i}) \\
&= R_i^k - c + J(k,i) - J(k,\bar{i})
\end{aligned}
$$

By using the lemma we have

$$R_i^k - 2c \leq Q_i^k \leq R_i^k.$$

Thus, if $R_i^k \leq 0$ then $Q_i^k \leq 0$ and it is optimal to stay. If $R_i^k \geq 2c$ then $Q_i^k \geq 0$ and it is optimal to switch.

3. (a)
   - The discretization method is straightforward to apply to all problems. There exist many good algorithms for nonlinear optimization. Drawbacks are the large number of variables and constraints, and that the solution may not converge to the solution of the original problem.
   - A shooting method is straightforward to apply to all problems, but it is crucial to find a good initial guess of $\lambda(0)$. The transition matrix may sometimes be ill conditioned when using a shooting method, but that is a minor problem for these quite simple problems.
   - A gradient method is straightforward to apply to (1) and (2), but (3) requires a slightly more complex gradient algorithm due to the the terminal constraint. Convergence tends to be slow for the gradient methods, but this is a minor problem for these quite simple problems.

(b) Problem (1) is a linear-quadratic problem that is possible to solve analytically with HJBE. Use $V(t,x) = P(t)x^2$, where $P(t)$ is a positive function that can be obtained by solving the Riccati equation. The optimal feedback law is $\mu(t,x) = -P(t)x$.

(c) Note that Problem (3) is the same problem as in exercise 1, but with an additional constraint on the final state. Thus, since S_f is a set with just one point then there is no constraint on $\lambda(t_f)$. Thus, the shape of the control signal can be derived, but with one unknown constant. By substituting the control signal in the dynamic model with the control law $u(t) = -1/2\lambda(t)$ we obtain a linear ODE of order one that is straightforward to solve and by using the initial and final state constraints all constants can be found.

4. (a) Since

$$\dot{q}(s) = q'(s)\dot{s},$$
$$\ddot{q}(s) = \frac{d}{dt}\big(q'(s)\big)\dot{s} + q'(s)\ddot{s} = q''(s)\dot{s}^2 + q'(s)\ddot{s},$$

it holds that

$$\begin{aligned}\tau &\overset{(1)}{=} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) \\ &= M\big(q(s)\big)\big(q''(s)\dot{s}^2 + q'(s)\ddot{s}\big) + C\big(q(s), q'(s)\dot{s}\big)q'(s)\dot{s} + G\big(q(s)\big) \\ &= M\big(q(s)\big)q'(s)\ddot{s} + \Big(M\big(q(s)\big)q''(s) + C\big(q(s), q'(s)\big)q'(s)\Big)\dot{s}^2 + G\big(q(s)\big), \\ &= m(s)\ddot{s} + c(s)\dot{s}^2 + g(s),\end{aligned}$$

where the third equality follows from the given information that $C(\cdot,\cdot)$ is linear in the joint velocities, i.e., $C\big(q(s), q'(s)\dot{s}\big) = C\big(q(s), q'(s)\big)\dot{s}$.

(b) We have

$$T = \int_0^T dt = \Big[dt = \frac{dt}{ds}ds = \frac{1}{\dot{s}}\,ds\Big] = \int_{s(0)}^{s(T)} \frac{1}{\dot{s}}\,ds = \int_0^1 \frac{1}{\dot{s}}\,ds,$$

which proves the first statement. Finally, since $\dot{b}(s) = b'(s)\dot{s}$, it holds that

$$b'(s) = \frac{1}{\dot{s}}\frac{d}{dt}\big(\dot{s}^2\big) = \frac{1}{\dot{s}}2(\dot{s})\ddot{s} = 2\ddot{s} \triangleq 2a(s).$$

(c) Now, the objective function can be describe by

$$T = \int_0^1 \frac{1}{\dot{s}}\,ds = \int_0^1 \frac{1}{\sqrt{b(s)}}\,ds,$$

and the constraints follows from the fact shown above and that

$$\begin{aligned}&b(0) = b(s(0)) = (\dot{s}(0))^2 = 0, \\ &b(1) = b(s(T)) = (\dot{s}(T))^2 = 0, \\ &b(s) = (\dot{s})^2 \geq 0.\end{aligned}$$

(d) The listed case yields

$$m(s) = 1, \quad c(s) = 0, \quad g(s) = \cos(s),$$

which in turn implies that

$$\tau = \ddot{s} + \cos(s) = a(s) + \cos(s).$$

Thus, the torque requirements can be stated as

$$-2 - \cos(s) \leq a(s) \leq 2 - \cos(s).$$

The Hamiltonian is given by

$$H(b, a, \lambda) \triangleq \frac{1}{\sqrt{b}} + 2\lambda a.$$

Pointwise minimization yields

$$a^*(s,\lambda) = \underset{-2-\cos(s)\leq a\leq 2-\cos(s)}{\arg\min}\left\{\frac{1}{\sqrt{b}} + 2\lambda a\right\} = \begin{cases} 2 - \cos(s), & \lambda < 0 \\ -2 - \cos(s), & \lambda > 0\end{cases}$$

The adjoint equation is given by

$$\lambda' = -\frac{\partial H}{\partial b}(b, a, \lambda) = \frac{1}{2}b^{-3/2}.$$

This implies that the switching function $\sigma = \lambda$ has the property

$$\sigma' = \lambda' = \frac{1}{2} b^{-3/2} > 0,$$

and thus at most one switching from $\lambda < 0$ to $\lambda > 0$ will occur. Now, it holds that

$$b'(s) = 2a(s) = \begin{cases} 4 - 2\cos(s), & \lambda < 0 \\ -4 - 2\cos(s), & \lambda > 0 \end{cases},$$

which implies that

$$b(s) = \begin{cases} 4s - 2\sin(s) + c_1, & \lambda < 0 \\ -4s - 2\sin(s) + c_2, & \lambda > 0 \end{cases},$$

for some constants c_1 and c_2. With $b(0) = b(1) = 0$ this becomes

$$b(s) = \begin{cases} 4s - 2\sin(s), & \lambda < 0 \\ -4s - 2\sin(s) + 4 + 2\sin(1), & \lambda > 0 \end{cases}.$$

The intersection

$$4s - 2\sin(s) = -4s - 2\sin(s) + 4 + 2\sin(1),$$

yields the switching time $\tilde{s} = \big(2 + \sin(1)\big)/4$ and thus

$$a^* = \begin{cases} 2 - \cos(s), & s < \tilde{s} \\ -2 - \cos(s), & s > \tilde{s} \end{cases}$$

is the optimal control policy.