EXAM IN OPTIMAL CONTROL (TSRT08)

ROOM: T2

TIME: Tuesday, August 22, 2023, 08.00–12.00

COURSE: TSRT08 OPTIMAL CONTROL

CODE: TEN1

DEPARTMENT: ISY

NUMBER OF EXERCISES: 4

NUMBER OF PAGES (including cover page): 7

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VISITS: 9 and 11

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APPROVED TOOLS: Formula sheet for the course, any collections of formulas and tables printed by a publishing house, calculator. No other books are allowed.

SOLUTIONS: Linked from the course home page after the exam.

The exam can be inspected and checked out 2023-09-12, Room 2A:573, B-building, entrance 25, A-corridor, to the right.

PRELIMINARY GRADING:	grade 3	15 points
	grade 4	23 points
	grade 5	30 points

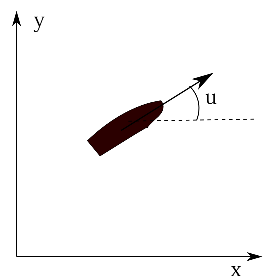
All solutions should be well motivated. Writing should be neat and clean.

Good Luck!

1. (a) A simple model of a boat moving at constant speed is

$$\dot{x} = V \cos u$$
$$\dot{y} = V \sin u$$

where x and y are the positions in the xy-plane, V is the constant speed and u is the heading angle.



It is desired to make a fishing trip from the initial position

$$x(0) = 0, \quad y(0) = 0$$

to the final position

$$x(1) = 1, \quad y(1) = 0$$

Since there is more fish at positions with higher y-coordinates the trip is planned to maximize

$$\int_0^1 y \; dt$$

Show that a fishing trip satisfying the Pontryagin minimum principle has the property that

$$\tan u = c_1 + c_2 t$$

for some constants c_1 and c_2. (5p)

(b) An industrial robot is configured to move a tool in one dimension. The position is x_1 and the velocity is x_2. Newton's force relation then gives the model

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u$$

where the applied force u is the control signal which is limited by

$$|u| \leq 1$$

One wishes to move the tool in such a way that it returns to its original position with maximum negative velocity, i.e. the optimization problem is

$$\min x_2(1), \text{ with } x_1(0) = 0, \quad x_2(0) = 0, \quad x_1(1) = 0$$

Compute the optimal open loop control u as a function of time. (5p)

2. A businessman operates out of a van that he sets up in one of two locations on each day. If he operates in location i (where $i = 1, 2$) on day k he makes a known and predictable profit denoted r_i^k. However, each time he moves from one location to the other, he pays a setup cost c. The businessman wants to maximize his total profit over N days.

   (a) The problem can be formulated as a shortest path problem (SPP) where the node (k, i) is representing location i at day k. Let s and e be the start node and the end node, respectively. The costs of all edges are:

   - s to i_1 with cost $-r_i^1$
   - i_k to i_{k+1} (i.e. no switch) with cost $-r_{i_{k+1}}^{k+1}, k = 1, ..., N-1$
   - i_k to $\bar{i}_{k+1}$ (i.e. switch) with cost $c - r_{\bar{i}_{k+1}}^{k+1}, k = 1, ..., N-1$
   - i_N to e with cost 0

   where $\bar{i}$ denotes the location that is not equal to i, i.e. $\bar{1} = 2$ and $\bar{2} = 1$. Write a figure to illustrate the SPP and the definitions of variables and parameters. Write the corresponding dynamic programming algorithm. (Note that you do not have to solve the problem.) (5p)

   (b) Suppose he is at location i on day $k-1$ and let

   $$R_i^k = r_{\bar{i}}^k - r_i^k.$$

   Show that if $R_i^k \leq 0$ it is optimal to stay at location i, while if $R_i^k \geq 2c$ it is optimal to switch. You can use the following lemma. *Lemma:* For every $k = 1, 2, ..., N$ it holds:

   $$|J(k, i) - J(k, \bar{i})| \leq c$$

   where $J(k, i)$ is the optimal cost-to-go function at stage k for state i. (5p)

3. Consider the following optimal control problems:

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & \int_0^{t_f} \left(x^2(t) + u^2(t) \right) \, dt \\ \text{subject to} \quad & \dot{x}(t) = x(t) + u(t), \\ & x(0) = 1 \end{aligned} \tag{1}$$

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & \int_0^{t_f} \left(x(t) + u^2(t) \right) \, dt \\ \text{subject to} \quad & \dot{x}(t) = x(t) + u(t) + 1, \\ & x(0) = 0 \end{aligned} \tag{2}$$

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & \int_0^{t_f} \left(x(t) + u^2(t) \right) \, dt \\ \text{subject to} \quad & \dot{x}(t) = x(t) + u(t) + 1, \\ & x(0) = 0 \\ & x(t_f) = 1. \end{aligned} \tag{3}$$

(a) Suppose you must solve these problems numerically. Describe advantages and disadvantages of (A) the discretization method (constrained nonlinear program), (B) the shooting method (boundary condition iteration), and (C) the gradient method (first order gradient search of the cost function) for solving these three optimal control problems. (5p)

(b) Make comments on if and how the problem (1) can be solved by using HJBE. Note that you do not necessarily have to solve the problems, but your statements must be well motivated. (2p)

(c) Make comments on if and how the problem (3) can be solved by using PMP. Note that you do not necessarily have to solve the problems, but your statements must be well motivated. (3p)

4. Consider the motion of a robotic manipulator with joint angles $q \in \mathbb{R}^n$, which may be described as a function of the applied joint torques $\tau \in \mathbb{R}^n$ as

$$\tau = M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q), \tag{$\star$}$$

where $M(q) \in \mathbb{R}^{n \times n}$ is a positive definite mass matrix and $C(q, \dot{q}) \in \mathbb{R}^{n \times n}$ is a matrix accounting for Coriolis and centrifugal effects, which is *linear* in the joint velocities, and where $G(q) \in \mathbb{R}^n$ is a vector accounting for gravity and other joint angle dependent torques.

Consider a path $q(s)$ as a function of a scalar path coordinate s. The path coordinate determines the spatial geometry of the path, whereas the trajectory's time dependency follows from the relation $s(t)$ between the path coordinate s and time t.

(a) Show that $(\star)$ can be expressed in terms of s as

$$\tau(s) = m(s)\ddot{s} + c(s)\dot{s}^2 + g(s),$$

where

$$\begin{aligned} m(s) &= M\big(q(s)\big)q'(s), \\ c(s) &= M\big(q(s)\big)q''(s) + C\big(q(s), q'(s)\big)q'(s), \\ g(s) &= G\big(q(s)\big), \end{aligned}$$

with the prime denoting derivative, i.e., $f'(x) = (df/dx)(x)$. (2p)

(b) Consider the time-optimal path tracking problem

$$\begin{aligned} \underset{s(\cdot)}{\text{minimize}} \quad & T \\ \text{subject to} \quad & \tau(s) = m(s)\ddot{s} + c(s)\dot{s}^2 + g(s), \\ & s(0) = 0, \\ & s(T) = 1, \\ & \dot{s}(0) = 0, \\ & s(\dot{T}) = 0, \\ & \underline{\tau}\big(s(t)\big) \leq \tau\big(s(t)\big) \leq \overline{\tau}\big(s(t)\big), \end{aligned}$$

where the torque lower bounds $\underline{\tau}$ and upper bounds $\overline{\tau}$ may depend on s. Using the fact that $dt = (dt/ds)ds$, show that

$$T = \int_0^1 \frac{1}{\dot{s}}\, ds,$$

and that for the change of variables

$$a(s) = \ddot{s},$$
$$b(s) = \dot{s}^2,$$

it holds that $b'(s) = 2a(s)$. (2p)

(c) Show that, the optimization problem in (b) is equivalent to

$$\begin{aligned}
\underset{a(\cdot)}{\text{minimize}} \quad & \int_0^1 \frac{1}{\sqrt{b(s)}}\, ds \\
\text{subject to} \quad & \tau(s) = m(s)a(s) + c(s)b(s) + g(s), \\
& b(0) = 0, \\
& b(T) = 0, \\
& b'(s) = 0, \\
& b'(s) = 2a(s), \\
& b(s) \geq 0, \\
& \underline{\tau} \leq \tau \leq \overline{\tau},
\end{aligned}$$

in the new coordinates. (1p)

(d) Compute the optimal path s, from the problem posed in (c) when

$$\begin{aligned}
q(s) &= s, \\
M(q) &= l^2 m = 1, \\
C(q, \dot{q}) &= 0, \\
G(q) &= mlg\cos(s) = \cos(s), \\
\underline{\tau}(s) &= -2, \quad \overline{\tau}(s) = 2.
\end{aligned}$$

Hint: Consider s as a “time-variable”. (5p)