EXAM IN OPTIMAL CONTROL (TSRT08)

ROOM: TER3

TIME: Wednesday, March 15, 2023, 08.00–12.00

COURSE: TSRT08 OPTIMAL CONTROL

CODE: TEN1

DEPARTMENT: ISY

NUMBER OF EXERCISES: 4

NUMBER OF PAGES (including cover page): 4

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VISITS: 9 and 11

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APPROVED TOOLS: Formula sheet for the course, any collections of formulas and tables printed by a publishing house, calculator. No other books are allowed.

SOLUTIONS: Linked from the course home page after the exam.

The exam can be inspected and checked out 2023-04-13, Room 2A:573, B-building, entrance 25, A-corridor, to the right.

PRELIMINARY GRADING:	grade 3	15 points
	grade 4	23 points
	grade 5	30 points

All solutions should be well motivated. Writing should be neat and clean.

Good Luck!

1. (a) Find the control signal $u(t)$ expressed as a function of the states $x(t)$ which satisfies the optimal control problem

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & x(T) + \int_0^T \frac{u^2(t)}{x(t)}\, dt \\ \text{subject to} \quad & \dot{x}(t) = -u(t), \end{aligned}$$

for a fixed T, using the PMP. (5p)

Hint: The adjoint equation is a separable ODE.

(b) Find the extremal to the functional

$$J(y) = \int_0^1 \left(e^t y + \frac{\dot{y}^2}{t} \right) dt,$$

satisfying $y(0) = 1$ and $y(1) = 0$. (5p)

2. (a) The system

$$\dot{x} = u$$

is controlled to minimize the criterion

$$\int_0^\infty (x^{2m} + u^2)\, dt$$

where m is a positive integer. Derive a control law $u = k(x)$ that minimizes the criterion. (5p)

(b) Consider an analogous discrete time problem. Minimize

$$\sum_0^{N-1} (x(t)^{2m} + u(t)^2) + x(N)^{2m}$$

for the system

$$x(t+1) = x(t) + u(t)$$

Write down the dynamic programming recursion for solving the problem. Explain why it is difficult to solve the problem explicitly if $m > 1$. (5p)

3. Consider the optimal control problem

$$\begin{array}{ll} \text{minimize} & \sum_{k=0}^{\infty} x_k^2 + u_k^2 \\ \text{subject to} & x_{k+1} = x_k + u_k \\ & x_0 \text{ given} \\ & u_k \in [-1, 1] \end{array}$$

(a) Compute an optimal feedback policy $u_k = \mu(x_k))$ for this problem when the control signal constraint is neglected. *Hint:* try the value function $J(x) = px^2$, where $p > 0$. (4p)

(b) Now, consider the case with constraints on the control signal. Compute an approximative solution by solving

$$\min_{-1 \leq u \leq 1} \left\{ x^2 + u^2 + J(x + u) \right\}$$

where $J(x)$ is defined as in the hint above. (3p)

(c) Prove that the closed loop system using the feedback of the previous subproblem is stable. (3p)

4. Consider the dynamical system

$$\frac{d^2y(t)}{dt^2} = u(t) + k$$

where $|u(t)| \leq 1$ is the control signal and where $0 < k < 1$ is a known constant. You should solve the optimal control problem

$$\underset{|u(t)|\leq 1}{\text{minimize}} \int_0^T |u(t)| dt$$

subject to the above dynamics with known initial values $y(0)$ and $\dot{y}(0)$ and such that $y(T) = 0$. The final time T should also be optimized.

(a) Use PMP to show that the optimal control signal is of the form

$$u(t) = \begin{cases} -1, & 0 \leq t \leq t_1 \\ 0, & t > t_1 \end{cases}$$

for some t_1. (4p)

(b) Show that the following feedback policy is optimal

$$u(y, \dot{y}) = \begin{cases} 0, & y \leq 0 \text{ or } y \leq \frac{\dot{y}^2}{2k} \text{ and } \dot{y} \leq 0 \\ -1, & \text{otherwise} \end{cases}$$

(4p)

(c) Consider the case where it is additionally required that $\dot{y}(T) = 0$. What form does the optimal control signal now have? (2p)