TSRT08: Optimal Control
Solutions

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1. (a) We have with $\dot{x} = dx/dy$ that $dx = \dot{x}dy$ and $\dot{y} = 1/\dot{x}$. This gives

$$\begin{aligned}\underset{y}{\text{minimize}} \int_0^L \frac{y\dot{y}^3}{1+\dot{y}^2} dx &= \underset{y}{\text{minimize}} \int_0^L \frac{y\frac{1}{\dot{x}^3}}{1+\frac{1}{\dot{x}^2}} dx \\ &= \underset{x}{\text{minimize}} \int_H^h \frac{y\frac{1}{\dot{x}^3}}{1+\frac{1}{\dot{x}^2}} \dot{x} dy \\ &= \underset{x}{\text{minimize}} \int_H^h \frac{y}{1+\dot{x}^2} dy\end{aligned}$$

(b) From the Euler-Lagrange equation we obtain the condition

$$\frac{d}{dy}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{y}{1+\dot{x}^2}\right)\right) = 0$$

and hence

$$\frac{\dot{x}y}{(1+\dot{x}^2)^2} = D \tag{1}$$

for some constant D. Since $p = -1/\dot{y}$ we obtain

$$y = -D\left(p + \frac{2}{p} + \frac{1}{p^3}\right)$$

Now we have $dy = -pdx = -p(dx/dp)dp$ and hence

$$\frac{dx}{dp} = -\frac{1}{p}\frac{dy}{dp} = D\left(\frac{1}{p} - \frac{2}{p^3} - \frac{3}{p^5}\right)$$

implying

$$x = C + D\left(\ln p + \frac{1}{p^2} + \frac{3}{4p^4}\right)$$

Alternative solution with PMP: Introducing the dynamics $\dot{x} = u$ gives the OCP

$$\begin{aligned}&\underset{x}{\text{minimize}} \int_H^h \frac{y}{1+u^2} dy \\ &\dot{x} = u\end{aligned} \tag{2}$$

The corresponding Hamiltonian is

$$H(y, x, u, \lambda) = \frac{y}{1+u^2} + \lambda u \tag{3}$$

and the adjoint equation $\dot{\lambda} = -\frac{\partial H}{\partial x} = 0$ implies that $\lambda = \tilde{D}$ for some constant $\tilde{D} \in \mathbb{R}$.

Now, a stationary point for the Hamiltonian w.r.t. u needs to satisfy

$$\frac{\partial H}{\partial u} = \frac{2yu}{(1+u^2)^2} + \lambda = 0 \Leftrightarrow \frac{2yu}{(1+u^2)^2} = -\tilde{D} \Leftrightarrow \frac{y\dot{x}}{(1+\dot{x}^2)^2} = D, \tag{4}$$

where we have used that $\lambda = \tilde{D}$ in the first equivalence and have recalled that $\dot{x} = u$ and have defined $D \triangleq -\frac{\tilde{D}}{2}$ in the last equivalence. This is the same as (1), so the rest of the solution is the same as the one above. Note: to rigorously show that a stationary point from above is, in fact, a minimizer to H requires additional arguments.

2. (a) By introducing the control $u = \dot{\theta}$, we can specify the optimal control problem

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & \int_{r_1}^{r_2} g(r)\sqrt{1+(r\cdot u(r))^2}dr \\ \text{subject to} \quad & \dot{\theta} = u \\ & \theta(r_1) = \theta_1 \\ & \theta(r_2) = \theta_2 \end{aligned}$$

where r_1, θ_1 and r_2, θ_2 are the polar coordinates of the points P_1 and P_2, respectively.

(b) With $g(r) = \alpha/r$, the Hamiltonian is given by

$$H(r,\theta,u,\lambda) = \frac{\alpha}{r}\sqrt{1+(r\cdot u)^2} + \lambda\cdot u$$

Further we have that

$$\begin{aligned} \frac{\partial H}{\partial u}(r,\theta,u,\lambda) &= \alpha\frac{r\cdot u}{\sqrt{1+(r\cdot u)^2}} + \lambda \\ \frac{\partial^2 H}{\partial u^2}(r,\theta,u,\lambda) &= \alpha\frac{r}{(1+(r\cdot u)^2)^{3/2}} \end{aligned}$$

Since $\alpha > 0$ and $r > 0$ we have that $\frac{\partial^2 H}{\partial u^2} > 0\ \forall u$. Hence, $H(r,\theta,u,\lambda)$ is strictly convex in u. Therefore, pointwise minimization yields

$$0 = \frac{\partial H}{\partial u}(r,\theta^*(r),u^*(r),\lambda) = \alpha\frac{r\cdot u^*(r)}{\sqrt{1+(r\cdot u^*(r))^2}} + \lambda(r)$$

The adjoint equation is given by

$$\dot{\lambda} = -\frac{\partial H}{\partial \theta}(r,\theta,u,\lambda) = 0$$

without final constraint on $\lambda(r_2)$ since we do have a final constraint on $\theta(r_2)$. This equation has the solution

$$\lambda(r) = c$$

for some constant c and the optimal control is

$$\alpha\frac{r\cdot u^*(r)}{\sqrt{1+(r\cdot u^*(r))^2}} = -c$$

This requires $r\cdot u^*(r)$ to be constant, which can be written as

$$ru^*(r) = a \quad \Rightarrow \quad u^*(r) = \frac{a}{r}$$

for some constant a. This gives the optimal path

$$\dot{\theta} = \frac{a}{r} \quad \Rightarrow \quad \theta = a\log r + b$$

which we were supposed to show.

(c) Reformulate the optimal path as a function of theta

$$r(\theta) = e^{\frac{\theta - b}{a}} = Be^{A\theta}$$

where $A = 1/a$ and $B = e^{-b/a}$. We now require that the initial and final point shall have the same radius r_0, such that $r(\theta_1) = r_0$ and $r(\theta_2) = r_0$. This gives

$$r_0 = Be^{A\theta_1}, \quad r_0 = Be^{A\theta_2} \quad \Rightarrow \quad A = 0, \quad B = r_0$$

which gives

$$r(\theta) = r_0, \quad \text{for} \quad \theta_1 \leq \theta \leq \theta_2$$

This corresponds to a path with constant radius, i.e. a circle segment.

3. The constraints $U(k, x_k)$ on the input signal may be expressed as

$$U(k, x_k) = \{u_k : ax_k^\alpha - u_k \geq 0\}.$$

The dynamic programming algorithm yields:

- $k = 2$: Then it holds that
$$J(2, x_2) = 0.$$

- $k = 1$: Now, the above yields
$$J(1, x_1) = \min_{u_1 \in U(1, x_1)} \{-\beta \log(u_1)\},$$
which takes its minimum at $u_1^* = ax_1^\alpha$. Thus,
$$J^*(1, x_1) = -\beta \log(ax_1^\alpha).$$

- $k = 0$: Finally, it follows that
$$\begin{aligned} J(0, x_0) &= \min_{u_0 \in U(0, x_0)} \{-\log(u_0) - \beta \log\left(a(ax_0^\alpha - u_0)^\alpha\right)\} \\ &= \min_{u_0 \in U(0, x_0)} \{\underbrace{-\log(u_0) - \beta \log(a) - \alpha\beta \log(ax_0^\alpha - u_0)}_{\triangleq h(u_0)}\}. \end{aligned}$$
The extrema are found via differentiation of h, i.e.,
$$0 = h'(u_0) = -\frac{1}{u_0} + \alpha\beta \frac{1}{ax_0^\alpha - u_0} = \frac{\alpha\beta u_0 - (ax_0^\alpha - u_0)}{u_0(ax_0^\alpha - u_0)} = \frac{(1 + \alpha\beta)u_0 - ax_0^\alpha}{u_0(ax_0^\alpha - u_0)}.$$
Since $1 + \alpha\beta > 1$, the extremum is attained by
$$u_0^* = \frac{1}{1 + \alpha\beta} ax_0^\alpha \in U(0, x_0),$$
with the corresponding optimal cost-to-go function easily calculated from the above. That this constitutes a minimum is verified by the fact that
$$h''(u_0) = \frac{1}{u_0^2} + \alpha\beta \frac{1}{(ax_0^\alpha - u_0)^2} \geq 0.$$

4. First consider the case when $x = 0$; the Bellman equation reads

$$\begin{aligned} V(0) &= \min_{u\in\{-1,0,1\}} \{u^2 + \gamma V(f(0,u))\} \\ &= \min\{(-1)^2 + \gamma V(f(0,-1)), \quad 0^2 + \gamma V(f(0,0)), \quad 1^2 + \gamma V(f(0,1))\} \\ &= \min\{1 + \gamma V(-1), \quad \gamma V(0), \quad 1 + \gamma V(1)\}, \end{aligned} \tag{5}$$

where we have used the given table for f in the last equality. Now, the middle case $V(0) = \gamma V(0)$ implies that $V(0) = 0$ since $\gamma \neq 1$. Since V is nonnegative this has to be the minimum (since the other cases will yield $V(0) > 1 > 0$). Hence, we have $V(0) = 0$ and the corresponding optimal action is $u = 0$.

Next, consider the case when $x = 1$; the Bellman equation reads

$$\begin{aligned} V(1) &= \min_{u\in\{-1,0,1\}} \{1^2 + u^2 + \gamma V(f(1,u))\} \\ &= \min\{1 + (-1)^2 + \gamma V(f(1,-1)), \quad 1 + 0^2 + \gamma V(f(1,0)), \quad 1 + 1^2 + \gamma V(f(1,1))\} \\ &= \min\{2 + \gamma V(0), \quad 1 + \gamma V(1), \quad 2 + \gamma V(1)\}. \end{aligned} \tag{6}$$

First of, the last case cannot be optimal since it is larger than the third case $(2+\gamma V(1) > 1+\gamma V(1))$. Therefore we have

$$V(1) = \min\{2 + \gamma V(0), \quad 1 + \gamma V(1)\} = \min\{2, \quad 1 + \gamma V(1)\}, \tag{7}$$

where $V(0) = 0$ from above have been used in the last equality. If the second case would be optimal we would get $V(1) = 1 + \gamma V(1) \Leftrightarrow V(1) = \frac{1}{1-\gamma}$. Hence,

$$V(1) = \min\left\{2, \quad 1 + \frac{\gamma}{1-\gamma}\right\} = \min\left\{2, \quad \frac{1}{1-\gamma}\right\}, \tag{8}$$

where the first case is the minimum if $2 \leq \frac{1}{1-\gamma} \Leftrightarrow \gamma \geq \frac{1}{2}$. Hence, we get $V(1) = 2$, with the corresponding control $u = -1$, if $\gamma \geq \frac{1}{2}$; and we get $V(1) = \frac{1}{1-\gamma}$, with the corresponding control $u = 0$, if $\gamma \leq \frac{1}{2}$.

A similar derivation for $x = -1$ gives $V(-1) = 2$, with the corresponding control $u = 1$, if $\gamma \geq \frac{1}{2}$, and $V(-1) = \frac{1}{1-\gamma}$, with the corresponding control $u = 0$, if $\gamma \leq \frac{1}{2}$.

In conclusion we get for $\gamma \geq \frac{1}{2}$ the optimal policy

$$\mu(x) = \begin{cases} 1, & \text{if } x = -1 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x = 1 \end{cases} \tag{9}$$

and for $\gamma \leq \frac{1}{2}$ we get the trivial policy $\mu(x) = 0$.