EXAM IN OPTIMAL CONTROL (TSRT08)

ROOM: TER2

TIME: Thursday, January 11, 2023, 14.00–18.00

COURSE: TSRT08 OPTIMAL CONTROL

CODE: TEN1

DEPARTMENT: ISY

NUMBER OF EXERCISES: 4

NUMBER OF PAGES (including cover page): 4

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VISITS: 15 and 17 by Daniel Arnstöm

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APPROVED TOOLS: Formula sheet for the course, any collections of formulas and tables printed by a publishing house, calculator. No other books are allowed.

SOLUTIONS: Linked from the course home page after the exam.

The exam can be inspected and checked out 2023-01-31, 12.30-13:00 in Ljungeln, B-building, entrance 27, A-corridor, to the right.

PRELIMINARY GRADING:	grade 3	15 points
	grade 4	23 points
	grade 5	30 points

All solutions should be well motivated. Writing should be neat and clean.

Good Luck!

1. Consider Newton's minimal resistance problem:

$$\text{minimize} \int_0^L \frac{y\dot{y}^3}{1+\dot{y}^2}dx$$

with variable y subject to $y(0) = H$ and $y(L) = h$, where $H > h$. Here $\dot{y} = dy/dx$.

   (a) Show that by considering x to be a function of y the following problem

   $$\text{minimize} \int_H^h \frac{y}{1+\dot{x}^2}dy$$

   is an equivalent problem to Newton's minimal resistance problem. Here $\dot{x} = dx/dy$. (3p)

   (b) Show that for the optimal solution it holds that x is related to $p = -\dot{y}$ as

   $$x = C + D\left(\ln p + \frac{1}{p^2} + \frac{3}{4p^4}\right)$$

   for some constants C and D. (7p)

2. We are interested in computing optimal transportation routes in a circular city. The cost for transportation per unit length is given by a function $g(r)$ that only depends on the radial distance r to the city center. This means that the total cost for transportation from a point P_1 to a point P_2 is given by

$$\int_{P_1}^{P_2} g(r)ds,$$

where s represents the arc length along the path of integration. In polar coordinates (θ, r) the total cost reads

$$\int_{P_1}^{P_2} g(r)\sqrt{1+(r\dot{\theta})^2}dr,$$

where $\theta = \theta(r)$, and $\dot{\theta} = d\theta/dr$.

(a) Formulate the problem of computing an optimal path as an optimal control problem. (2p)

(b) For the case of $g(r) = \alpha/r$ for some positive α show that any optimal path satisfies the equation $\theta = a \ln r + b$ for some constants a and b. (5p)

(c) Show that if the initial point and the final point are at the same distance from the origin, then the optimal path is a circle segment. You may use the claim in (b). (3p)

3. Consider the problem

$$\begin{aligned} \text{maximize} \quad & \sum_{k=0}^{N-1} \beta^k \log(u_k) \\ \text{subject to} \quad & x_{k+1} = ax_k^{\alpha} - u_k \geq 0, \\ & x_0 \geq 0 \text{ given}, \end{aligned}$$

where $0 < \alpha, \beta < 1$ and $a > 0$ are some constants. This is commonly referred to as the *consumption problem* in the theory of economics. The variable u_k may be interpreted as the consumption for time period k and x_k the available capital, which is assumed positive, at time period k, respectively. Find the optimal control signal u_k, where $k = 1, 0$, for the problem when the horizon is $N = 2$ using the dynamic programming algorithm. (10p)

4. Let $\mathcal{D} = \mathcal{E} = \{-1, 0, 1\}$ and let $f : \mathcal{D}^n \times \mathcal{E} \to \mathcal{D}$ and $f_0 : \mathcal{D} \times \mathcal{E} \to \mathbf{R}$ be functions, where $f_0(x, u) = x^2 + u^2$ and where f is defined via the table

$x \backslash u$	-1	0	1
-1	-1	-1	0
0	-1	0	1
1	0	1	1

Consider the infinite horizon optimal control problem

$$\begin{aligned} \text{minimize } & \sum_{k=0}^{\infty} \gamma^k f_0(x_k, u_k) \\ \text{subject to } & x_{k+1} = f(x_k, u_k), \quad k \geq 0 \end{aligned}$$

with variables $(u_0, x_1, u_1, x_2 \ldots)$, where x_0 is given, and where $0 < \gamma < 1$ is a discount factor.

The Bellman equation for this optimal control problem is given by

$$V(x) = \min_{u \in \mathcal{E}} \{f_0(x, u) + \gamma V(f(x, u))\}$$

Compute the optimal solution to the optimal control problem.

Hint: You will get different answers for the case when $\gamma \leq 1/2$ and the case when $\gamma > 1/2$. (10p)