TSRT08: Optimal Control
Solutions

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1. (a) The Hamiltonian is given by

$$H(t, x, u, \lambda) = x + u^2 + \lambda(x + u + 1).$$

Pointwise minimization yields

$$0 = \frac{\partial H}{\partial u}(t, x, u, \lambda) = 2u + \lambda \quad \Rightarrow \quad u^* = -\frac{1}{2}\lambda.$$

The adjoint equation is given by

$$\dot{\lambda}(t) = -\lambda(t) - 1, \quad \lambda(T) = 0$$

which is a first order linear ODE with the solution

$$\lambda(t) = e^{T-t} - 1,$$

and the optimal control is

$$u^*(t) = -\frac{1}{2}\lambda(t) = \frac{1 - e^{T-t}}{2}.$$

(b) Introducing $x = y$ and $u = \dot{y}$, it holds that $\dot{x} = u$ and the Hamiltonian is given by

$$H(t, x, u, \lambda) = x^2 + u^2 + \lambda u.$$

The following equations must hold

$$\dot{\lambda} = -\frac{\partial H}{\partial x}(t, x, u, \lambda) = -2x,$$
$$0 = \frac{\partial H}{\partial u}(t, x, u, \lambda) = 2u + \lambda.$$

The latter yields $\dot{\lambda} = -2\dot{u} = -2\ddot{y}$, which plugged into the first equation gives

$$-2\ddot{y} = -2x = -2y \quad \Leftrightarrow \quad \ddot{y} - y = 0,$$

which has the solution

$$y(t) = c_1 e^t + c_2 e^{-t},$$

for some constants c_1 and c_2. The boundary constraints $y(0) = 0$ and $y(1) = 1$ yields

$$\begin{pmatrix} 1 & 1 \\ e^1 & e^{-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which has the solution

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{e^{-1} - e^1} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Thus,

$$y(t) = \frac{1}{e^{-1} - e^1}(e^{-t} - e^t),$$

is the sought after extremal.

2. (a) The cost function that we are considering can be expressed as

$$J = \int_0^T 1 \, dt$$

This gives the following Hamilton function

$$H(x, u, \lambda) = 1 + \lambda_1 x_2 + \lambda_2 x_3 + \lambda_3 u$$

The adjoint equations will then be

$$\begin{aligned}
\dot{\lambda}_1(t) &= -\frac{\partial H}{\partial x_1} = 0 \\
\dot{\lambda}_2(t) &= -\frac{\partial H}{\partial x_2} = -\lambda_1(t) \\
\dot{\lambda}_3(t) &= -\frac{\partial H}{\partial x_3} = -\lambda_2(t).
\end{aligned}$$

This gives $\lambda_1(t) = C_1$, $\lambda_2(t) = -C_1 t + C_2$ and $\lambda_3(t) = -\frac{C_1}{2}t^2 + C_2 t + C_3$. Since both $x(0)$ and $x(T)$ are fixed, we will not have any boundary conditions on $\lambda(t)$.
Furthermore, we have that

$$u = \arg \min_{-1 \leq u \leq 1} H(x^*, u, \lambda) = -\text{sign } \lambda_3 = \text{sign } (-\lambda_3)$$

Thus, the necessary condition for optimality is then satisfied by the control

$$u(t) = \text{sign } p(t)$$

where $p(t) = -\lambda_3(t) = \frac{C_1}{2}t^2 - C_2 t - C_3$, which is a polynomial with maximum degree of two. Since a second order polynomial can change sign at most two times, also the control will change sign at most two times.

(b) The Hamiltonian for the problem is

$$H(u(t), x(t), \lambda(t)) = |u(t)| + \lambda_1(t) x_2(t) + \lambda_2(t) u(t)$$

and the terminal cost

$$\phi(x) = x_1^2$$

The adjoint equations will then be

$$\begin{aligned}
\dot{\lambda}_1(t) &= -\frac{\partial H}{\partial x_1} = 0 \\
\dot{\lambda}_2(t) &= -\frac{\partial H}{\partial x_2} = -\lambda_1(t).
\end{aligned}$$

This gives $\lambda_1(t) = C_1$ and $\lambda_2(t) = -C_1 t + C_2$. Furthermore, with the boundary constraints we get

$$\begin{aligned}
\lambda_1(1) &= \frac{\partial \phi}{\partial x_1}(x^*(1)) = 2x_1^*(1) = C_1 \\
\lambda_2(1) &= \frac{\partial \phi}{\partial x_2}(x^*(1)) = 0
\end{aligned}$$

The control signal is chosen as

$$u = \arg \min_{-1 \leq u \leq 1} H(x^*, u, \lambda) = \begin{cases} -1 & \lambda_2 > 1 \\ 0 & |\lambda_2| \leq 1 \\ 1 & \lambda_2 < -1 \end{cases}$$

Since λ_2 is a linear function which ends at $\lambda_2(1) = 0$, u can change value at maximum one time, either from -1 to 0 or from 1 to 0.

3. (a) We have the optimization problem on standard form with $N = 5$, $\phi(x) = 0$, $f_0(k, x, u) = \upsilon_k u$, and $f(k, x, u) = x + \omega_k u$. Note that $x_k + \omega_k u_k \leq 10 \iff u_k \leq (10 - x_k)/\omega_k$.

Stage $k = N + 1 = 6$: $J(6, x) = 0$.

Stage $k = 5$:

$$\begin{aligned} J(5,x) &= \max_{0 \leq u \leq (W-x)/\omega_5, u \in \{0,1\}} \{\upsilon_5 u + J(6, x + \omega_5 u)\} \\ &= \max_{0 \leq u \leq (10-x)/2, u \in \{0,1\}} \{3u\} \end{aligned} \tag{1}$$

since the optimal control u and the expression inside the brackets vary for different x ,

x	0	1	2	3	4	5	6	7	8	9	10
$J(5,x)$	3	3	3	3	3	3	3	3	3	0	0
$\mu(5,x)$	1	1	1	1	1	1	1	1	1	0	0

Stage $k = 4$:

$$\begin{aligned} J(4,x) &= \max_{0 \leq u \leq (W-x)/\omega_4, u \in \{0,1\}} \{\upsilon_4 u + J(5, x + \omega_4 u\} \\ &= \max_{0 \leq u \leq (10-x)/3, u \in \{0,1\}} \{u + J(5, x + 3u)\} \end{aligned} \tag{2}$$

x	0	1	2	3	4	5	6	7	8	9	10
$J(4,x)$	4	4	4	4	4	4	3	3	3	0	0
$\mu(4,x)$	1	1	1	1	1	1	0	0	0	0	0

Stage $k = 3$:

$$\begin{aligned} J(3,x) &= \max_{0 \leq u \leq (W-x)/\omega_3, u \in \{0,1\}} \{\upsilon_3 u + J(4, x + \omega_3 u\} \\ &= \max_{0 \leq u \leq (10-x)/4, u \in \{0,1\}} \{7u + J(4, x + 4u)\} \end{aligned} \tag{3}$$

x	0	1	2	3	4	5	6	7	8	9	10
$J(3,x)$	11	11	10	10	10	7	7	3	3	0	0
$\mu(3,x)$	1	1	1	1	1	1	1	0	0	0	0

Stage $k = 2$:

$$\begin{aligned} J(2,x) &= \max_{0 \leq u \leq (W-x)/\omega_2, u \in \{0,1\}} \{\upsilon_2 u + J(3, x + \omega_2 u\} \\ &= \max_{0 \leq u \leq (10-x)/5, u \in \{0,1\}} \{8u + J(3, x + 5u)\} \end{aligned} \tag{4}$$

x	0	1	2	3	4	5	6	7	8	9	10
$J(2,x)$	15	15	11	11	10	8	7	3	3	0	0
$\mu(2,x)$	1	1	1	1	0	1	0	0	0	0	0

Stage $k = 1$:

$$\begin{aligned} J(1,x) &= \max_{0 \leq u \leq (W-x)/\omega_1, u \in \{0,1\}} \{\upsilon_1 u + J(2, x + \omega_1 u\} \\ &= \max_{0 \leq u \leq 10-x, u \in \{0,1\}} \{2u + J(2, x + u)\} \end{aligned} \tag{5}$$

x	0	1	2	3	4	5	6	7	8	9	10
$J(1,x)$	17	15	13	12	10	9	9	5	3	2	0
$\mu(1,x)$	1	0	1	1	0/1	1	0	1	0	1	0

The maximum is achieved for,

$$J^*(1,x) = 17 \tag{6}$$

(b) Using the dynamics,

$$\begin{aligned} x_{k+1} &= x_k + \omega_k u_k, \\ x_2 &= 0 + 1 = 1 \Rightarrow J(2,1) = 15, u_2 = 1 \\ x_3 &= 1 + 5 = 6 \Rightarrow J(3,6) = 7, u_3 = 1 \\ x_4 &= 6 + 4 = 10 \Rightarrow J(4,10) = 0, u_4 = 0 \\ x_5 &= 10 + 0 = 10 \Rightarrow J(5,10) = 0, u_5 = 0 \end{aligned}$$

In summary the knapsack is loaded optimally as follows,

k	x_k	J_k	u_k
1	0	17	1
2	1	15	1
3	6	7	1
4	10	0	0
5	10	0	0

4. (a) The Bellman equation is given by

$$J(x) = \min_u \{f_0(x,u) + J(f(x,u))\} \tag{7}$$

where,

$$\begin{aligned} f_0(x,u) &= \rho x^2 + u^2, \\ f(x,u) &= x + u \end{aligned}$$

Assume that the cost is on the form $J = px^2,\ p > 0$. Then the optimal control law is obtained by minimizing (**??**)

$$\begin{aligned} \mu(x) &= \arg\min_u \{f_0(x,u) + V(f(x,u))\}, \\ &= \arg\min_u \{\underbrace{\rho x^2 + u^2 + p(x+u)^2}_{H}\}, \\ H_u &= 2u + 2p(x+u), H_{uu} = 2(1+p) \succ 0 \end{aligned}$$

Therefore, $H_u = 0$ gives the minimum,

$$u^* = -p/(1+p)\ x$$

Inserting u^* in (10),

$$px^2 = \rho x^2 + p^2/(1+p)^2 x^2 + px^2(1 - p/(1+p))^2 = (\rho + p/(1+p))x^2$$

For this to hold for all x,

$$p = \rho + p/(1+p)$$

which has the positive solution

$$p = (\rho + \sqrt{\rho^2 + 4\rho})/2$$

(b) Since $J_0(x) = 0$, it follows that,

$$J_0(x) = p_0 x^2 \text{ with } p_0 = 0$$

Assume that $J_k(x) = p_k x^2$,then

$$J_{k+1}(x) = \min_u \{f_0(x,u) + J_k(f(x,u))\} = \min_u \{\rho x^2 + u^2 + p_k(x+u)^2\}$$

As in (a), the minimum is obtained when the gradient of the expression in the parenthesis is zero, i.e.

$$2u + 2p_k(x+u) = 0$$

resulting in

$$u = -p_k/(p_k+1)x = l_k x$$

Back-substituting gives

$$J_{k+1}(x) = (\rho + \frac{p_k}{p_k+1})x^2$$

Hence, $J_{k+1}(x) = p_{k+1}x^2$ if

$$p_{k+1} = \rho + \frac{p_k}{p_k+1}$$

(c) From(i) it follows that,

$$p_k x^2 = \rho x^2 + (l_k x)^2 + p_k(x + l_k x)^2$$

and since this has to hold for all x, it follows that,

$$p_k = \rho + l_k^2 + p_k(1+l_k)^2$$

Solving for p_k results in the desired recursion,

$$p_k = \frac{\rho + l_k^2}{1-(1+l_k)^2}$$

From (ii) it follows that,

$$\begin{aligned}\mu_{k+1}(x) &= \arg\min_u \{f_0(x,u) + J_k(f(x,u))\} \\ &= \arg\min_u \{\rho x^2 + u^2 + p_k(x+u)^2\}\end{aligned}$$

which is similar to the minimization in (b)and hence,

$$\begin{aligned}\mu_{k+1}(x) &= \frac{-p_k}{p_k+1}x \\ l_{k+1} &= \frac{-p_k}{p_k+1}\end{aligned}$$

(d) According to the following table, the policy-iteration method (c), converges much faster than the value-iteration method (b).

k	$p_k(b)$	$l_k(b)$	$p_k(c)$	$l_k(c)$
0	0	0	2.6842	-0.1000
1	0.5000	-0.3333	1.1128	-0.7286
2	0.8333	-0.4545	1.0018	-0.5267
3	0.9545	-0.4884	1.0000	-0.5005
4	0.9884	-0.4971	1.0000	-0.5000
5	0.9971	-0.4993	1.0000	-0.5000