EXAM IN OPTIMAL CONTROL

TIME: March 16, 2022, 14 – 18

COURSE: TSRT08, Optimal Control

PROVKOD: TEN1

DEPARTMENT: ISY

NUMBER OF EXERCISES: 4

NUMBER OF PAGES (including cover pages): 5

RESPONSIBLE TEACHER: Anders Hansson, phone 070–3004401

VISITS: Approximately 15,00, and 16.30

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APPROVED TOOLS: Formula sheet for the course, printed collections of formulas and tables, calculator. No other books are allowed.

SOLUTIONS: Linked from the course home page after the examination.

PRELIMINARY GRADING:	grade 3	15 points
	grade 4	23 points
	grade 5	30 points

All solutions should be well motivated. Writing should be neat and clean. Every page should be marked in such a way that your identity can be clearly identified. Scanning of pages should be done in such a way that it is easy to read the pages.

Good Luck!

1. (a) Solve the optimal control problem

$$\begin{aligned}
\underset{u(\cdot)}{\text{minimize}} \quad & \int_0^T \big(x(t) + u^2(t)\big)\, dt \\
\text{subject to} \quad & \dot{x}(t) = x(t) + u(t) + 1, \\
& x(0) = 0.
\end{aligned}$$

for a fixed $T > 0$, using the PMP. (5p)

(b) Find the extremal to the functional

$$J(y) = \int_0^1 \big(y^2(t) + \dot{y}^2(t)\big)\, dt,$$

satisfying $y(0) = 0$ and $y(1) = 1$. (5p)

2. (a) Consider controlling the triple integrator

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= u, \qquad |u| \leq 1
\end{aligned}$$

from a certain arbitrary initial condition $x(0)$ to a final value $x(T) = 0$ in minimum time. Show that the necessary conditions for optimality are satisfied by a control of the form

$$u(t) = \operatorname{sign} p(t)$$

where $p(t)$ is a polynomial. What is the maximum degree of the polynomial? How many times can u change sign? It is not necessary to compute the values of the coefficients of $p(t)$. (5p)

(b) Consider the system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u
\end{aligned}$$

where the control signal is limited by $|u| \leq 1$. One wants to minimize the criterion

$$\int_0^1 |u|\, dt + x_1(1)^2$$

for a given initial value. Show that the necessary conditions for optimality are of the type "bang-zero-bang", i.e. u changes finitely many times between -1, 0 and 1. What is the maximum number of switches? (5p)

3. Consider the problem of packing a knapsack with items labeled $k \in \{1, 2, \dots, N\}$ of different value v_k and weight w_k. There is only one copy of each item, and the task is to pack the knapsack such that the sum of the value of the items in the knapsack is maximized subject to the constraint that the total weight of the items is less than or equal to the weight limit W. The problem can be posed as a multi-stage decision problem in which at each stage k it is decided whether item k should be loaded or not. This decision can be coded in terms of the binary variable $u_k \in \{0, 1\}$, where $u_k = 1$ in case item k is loaded and $u_k = 0$ otherwise. If x_k denotes the total weight of the knapsack after the first $k-1$ items have been loaded, then the following relation holds:

$$x_{k+1} = x_k + w_k u_k, \qquad , x_1 = 0$$

for $k = 1, 2, \dots, N$. The constraint that $x_{k+1} \leq W$ can be reformulated in terms of u_k as $u_k \leq (W - x_k)/w_k$. From this it follows that it is possible to calculate how to load the knapsack in an optimal way using the dynamic programming recursion

$$J(n, x) = \max_{u \leq (W-x)/w_n, u \in \{0,1\}} \{v_n u + J(n+1, x + w_n u)\}$$

with final value $J(N+1, x) = 0$. We will consider the case when $W = 10$ and when the values of v_k and w_k are defined as in the table below, where $N = 5$.

k	1	2	3	4	5
v_k	2	8	7	1	3
w_k	1	5	4	3	2

For this case we notice that $x_k \in \{0, 1, \dots, 10\}$.

(a) Compute the values of $J(n, x)$ and the maximizing argument $u = \mu(n, x)$ for each value of $n = 1, \dots N$ as a table in terms of the values of x. The tables should look like

x	0	1	2	3	4	5	6	7	8	9	10
$J(n, x)$											
$\mu(n, x)$											

(6 p)

(b) From the tables derived in (a) compute the optimal loading of the knapsack. (4p)

4. Consider the infinite time horizon optimal control problem

$$\begin{array}{ll} \text{minimize} & \sum_{k=0}^{\infty} f_0(x_k, u_k) \\ \text{subject to} & x_{k+1} = f(x_k, u_k), \end{array}$$

with given initial vale x_0. You may assume that $f_0(x,u) = \rho x^2 + u^2$ and $f(x,u) = x + u$, where x and u are scalar-valued, and where $\rho > 0$.

(a) Compute the optimal feedback policy using the Bellman equation by guessing that $J(x) = px^2$ for some p. (2p)

(b) In general it is more tricky to solve the Bellman equation and different iterative procedures are available. One is so-called value-iteration, where $J_k(x)$ are iteratively computed from

$$J_{k+1}(x) = \min_u \{f_0(x,u) + J_k(f(x,u))\}$$

for $k = 1, 2, \ldots$ with $J_0(x) = 0$. Show that $J_k(x) = p_k x^2$ and that the minimizing argument is $l_k x_k$, where $l_k = -p_k/(p_k + 1)$ and where

$$p_{k+1} = \rho + p_k/(p_k + 1), \qquad p_0 = 0$$

(3p)

(c) There is an alternative procedure called policy iteration. In this method one starts with an initial feedback policy $\mu_0(x)$ and repeats the following two steps iteratively for $k = 0, 1, 2, \ldots$:

i. Compute $J_k(x)$ such that

$$J_k(x) = f_0(x, \mu_k(x)) + J_k(f(x, \mu_k(x)))$$

ii. Compute $\mu_{k+1}(x)$ as the minimizing argument in

$$\min_u \{f_0(x,u) + J_k(f(x,u))\}$$

Assume that $\mu_0(x) = l_0 x$ and that $J_k(x) = p_k x^2$. Show that $\mu_{k+1}(x) = l_{k+1} x$, where now

$$p_k = \frac{\rho + l_k^2}{1 - (1 + l_k)^2}$$

and

$$l_{k+1} = -p_k/(p_k + 1)$$

with l_0 given.

(3p)

(d) Compute the sequences p_k and l_k in (b) and (c) for $k = 1, 2, \ldots, 5$ when $\rho = 0.5$. Assume that $l_0 = -0.1$ for the method in (b). The iterates will converge to the solution in (a). Which method converges the fastest? (2p)