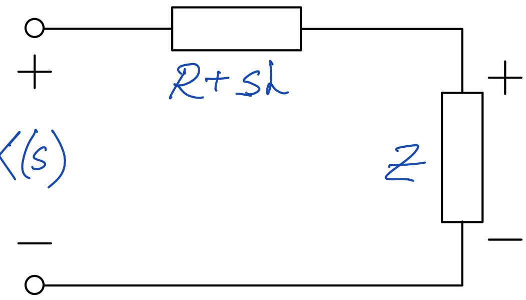
LÖSNINGAR, TSDT18/84 Signaler & System 2024-01-10

1 a)

Operatorschema:



Spänningsdelning

$$Y(s) = X(s) \cdot \frac{Z}{Z + (R + sL)}$$

(Utgå från energifritt system $\Rightarrow Y(s) = Y_{zs}(s)$)

där $Z = \frac{1}{sC} // R = \frac{\frac{1}{sC} \cdot R}{\frac{1}{sC} + R} = \frac{R}{sRC+1}$

$$\Rightarrow H(s) = \frac{Y_{zs}(s)}{X(s)} = \frac{\frac{R}{sRC+1}}{\frac{R}{sRC+1} + R + sL}$$

$$= \frac{\frac{1}{LC}}{s^2 + \left(\frac{R}{L} + \frac{1}{RC}\right)s + \frac{2}{LC}} = \left| \begin{matrix}\text{Formels.} \\ \text{sid. 13,} \\ N=2\end{matrix} \right| = \frac{\frac{1}{2} \cdot \omega_p^2}{s^2 + a_1 \cdot s + \omega_p^2} \quad \star$$

Butterworthfilter är vanligen amplitudnormerade $\Rightarrow |H(\omega)|_{max} = |H(0)| = 1$.
Här behöver vi dock justera nivåkonstanten med en faktor $\frac{1}{2}$ eftersom $H(0) = \frac{1}{2}$, enligt uppgift.

$\star \Rightarrow \omega_p^2 = \frac{2}{LC}$ (alt. $\frac{1}{2} \cdot \omega_p^2 = \frac{1}{LC}$)

$$\Rightarrow \underline{C} = \frac{2}{L \cdot \omega_p^2} = \frac{2}{\frac{\sqrt{2}}{6}(6 \cdot 10^3)^2} = \underline{\frac{\sqrt{2}}{6} \cdot 10^{-6}\, F} \quad \left(\frac{\sqrt{2}}{6}\, \mu F\right)$$

---

b) Butterworthfiltret är stabilt (ty passivt elektriskt nät)

$\Rightarrow$ en insignal $x(t) = \sin(2\omega_p t)$ ger upphov till en utsignal

$y(t) = A \cdot \sin(2\omega_p t + \arg H(2\omega_p))$,

där $\underline{A} = |H(2\omega_p)| = \left| \begin{matrix}\text{Formels. sid. 13,} \\ N=2 \ \& \ \omega = 2\omega_p\end{matrix} \right| = \frac{1}{2} \cdot \frac{1}{\sqrt{1 + \left(\frac{2\omega_p}{\omega_p}\right)^{2 \cdot 2}}} = \underline{\frac{1}{2\sqrt{17}}}$ ($\approx 0{,}12$)

Samma justering som i a)

② a) $H(\omega) = H(s)\big|_{s=j\omega}$ om systemet är stabilt, där $H(s)$ kan erhållas från

$g(t) = /\text{LTI-system}/ = (u * h)(t) \iff G(s) = U(s)\cdot H(s)$

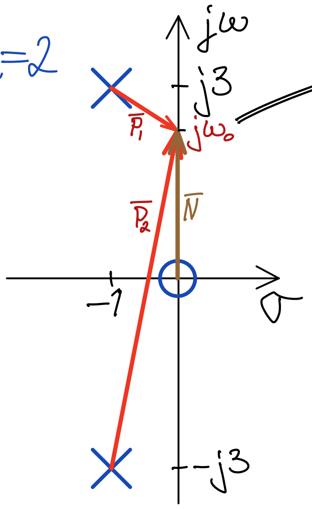
$$\Rightarrow \underline{H(s)} = \frac{G(s)}{U(s)} = \left/ \begin{array}{l} \text{Tab. 5:21} \Rightarrow G(s) = \frac{2}{3}\cdot\frac{3}{(s+1)^2+3^2}\ ;\ Re\{s\} > -1 \\ \text{Tab. 5:3} \Rightarrow U(s) = \frac{1}{s}\ ;\ Re\{s\} > 0 \end{array} \right/$$

$$= \underline{\frac{2s}{(s+1)^2+3^2}\ ;\ Re\{s\} > -1}$$

(Anm: $h(t) = \frac{dg(t)}{dt} \longrightarrow H(s)$ går också, men är krångligare)

Pol-nollställediagram för $H(s)$ — med pol-nollställevektorer:

$K=2$

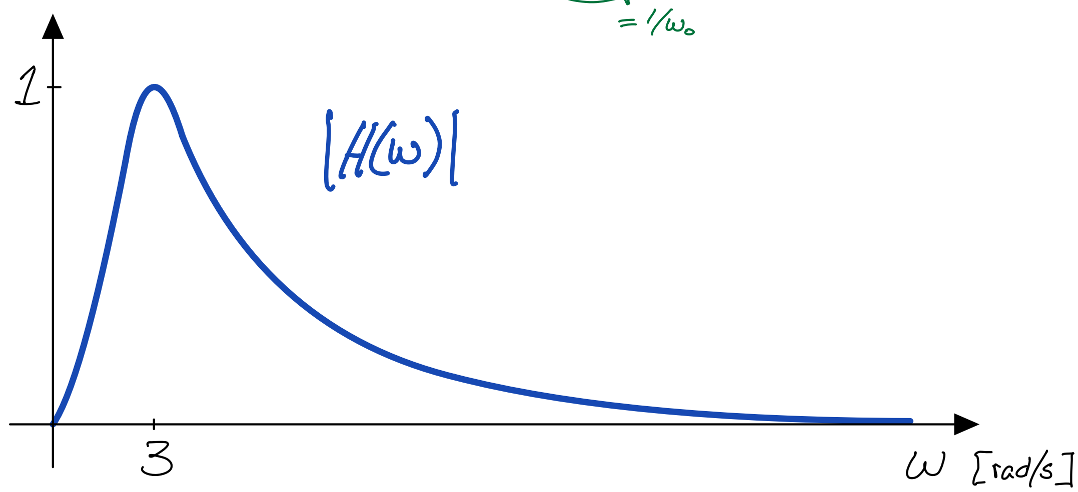


$$\underline{|H(\omega_0)| = |K|\cdot\frac{|\overline{N}|}{|\overline{P}_1|\cdot|\overline{P}_2|}}$$

- $|H(0)| = 2\cdot\frac{0}{\sqrt{1^2+3^2}\cdot\sqrt{1^2+3^2}} = 0$
- $\omega_0 = 3$ rad/s, dvs i höjd med den övre polen ⇒ lokalt max vid $\omega_0 \approx 3$:

  $$|H(3)| = 2\cdot\frac{3}{1\cdot\sqrt{1^2+6^2}} = \frac{6}{\sqrt{37}} \approx 1$$
- $\lim_{\omega_0\to\infty} |H(\omega_0)| \approx \lim_{\omega_0\to\infty} 2\cdot\underbrace{\frac{\omega_0}{\omega_0\cdot\omega_0}}_{=1/\omega_0} = 0$

⇒



---

b) Spärrband för låga vinkelfrekvenser ($\omega \to 0$) och för höga vinkelfrekvenser ($\omega \to \infty$). Passband ($|H(\omega)| > \frac{1}{\sqrt{2}}|H(\omega)|_{max}$) med mitt-vinkelfrekvens $\omega \approx 3$ rad/s ⇒ Det är ett bandpassfilter (BP).

③ a) Periodisk insignal $x_a(t)=\sum_{n=-\infty}^{\infty} D_n\cdot e^{jn\omega_0 t}$, där $\begin{cases} D_n=\dfrac{j4}{n+j} \\ \omega_0=5\text{ rad/s}\end{cases}$

$h(t)=3e^{-2t}u(t)$ är absolutintegrerbar $\Rightarrow$ systemet är stabilt

$$\Rightarrow\ y_a(t)=\sum_{n=-\infty}^{\infty}\hat{D}_n\cdot e^{jn\omega_0 t},\ \text{där}\ \hat{D}_n=D_n\cdot H(n\cdot\omega_0)$$

där $H(\omega)=\mathcal{F}\{3e^{-2t}u(t)\}=\big/\text{Formels. Tab. 3:5}\big/=\dfrac{3}{2+j\omega}\ \Rightarrow\ H(n\omega_0)\overset{\omega_0=5\text{ rad/s}}{=}\dfrac{3}{2+j5n}$

Svar: $y_a(t)=\sum_{n=-\infty}^{\infty}\hat{D}_n\cdot e^{jn\omega_0 t}$, där $\hat{D}_n=\dfrac{j4}{n+j}\cdot\dfrac{3}{2+j5n}=\dfrac{j12}{-3n+j(2+5n^2)}$

och $\omega_0=5$ rad/s $\quad\left(=\dfrac{12}{2+5n^2+j3n}\right)$

---

b) i) LTI-system $\Rightarrow y_b(t)=(x_b*h)(t)$. Stabilt system $\Rightarrow$

$$Y_b(\omega)=X_b(\omega)H(\omega)=\frac{2}{3-j\omega}\cdot\underbrace{\frac{3}{2+j\omega}}_{\text{från a)}}=\Big/\text{P.B.U.; Enklast om "}j\omega\text{" byts mot t.ex. "}s\text{"}\Big/$$

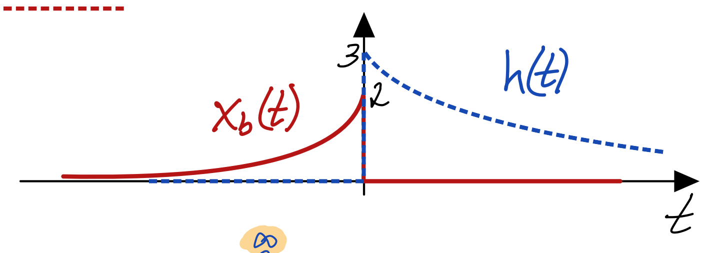
$$=\frac{6}{5}\left(\frac{1}{3-j}+\frac{1}{2+j\omega}\right)$$

Formels. Tab. 3:6 & 3:5 $\Rightarrow$ $y_b(t)=\frac{6}{5}\left(e^{3t}u_0(-t)+e^{-2t}u(t)\right)$

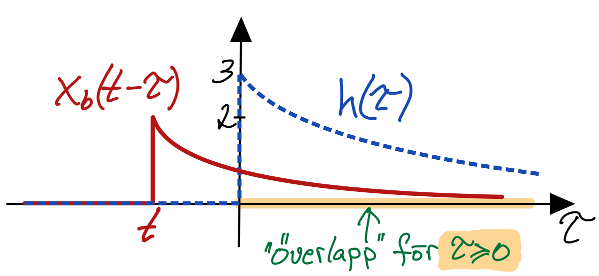
---

ii) $y_b(t)=(x_b*h)(t)=\int_{-\infty}^{\infty}x_b(t-\tau)h(\tau)\,d\tau$ $\quad$ (eller $\int_{-\infty}^{\infty}x(\tau)h(t-\tau)\,d\tau$)

Tab. 3:6 $\Rightarrow$ $x_b(t)=2e^{3t}u_0(-t)$

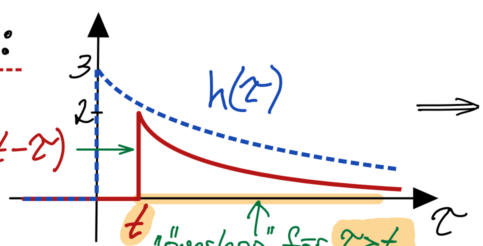


$t<0$:



$$\Rightarrow\ y_b(t)=\int_0^{\infty}2e^{3(t-\tau)}\cdot 3e^{-2\tau}\,d\tau=6e^{3t}\int_0^{\infty}e^{-5\tau}\,d\tau=\frac{6}{5}e^{3t}$$

$t\geq 0$:



$$\Rightarrow\ y_b(t)=6e^{3t}\int_t^{\infty}e^{-5\tau}\,d\tau=\frac{6}{5}e^{-2t}$$

Svar: $y_b(t)=\frac{6}{5}\left(e^{3t}u_0(-t)+e^{-2t}u(t)\right)$

4 a) Grafen $\Rightarrow$ $h[n]=\frac{1}{3}(u[n]-u[n-3]) = \frac{1}{3}(\delta[n]+\delta[n-1]+\delta[n-2])$

LTI-system $\Rightarrow$

$$\underline{g[n]} = u[n]*h[n] = \frac{1}{3}(u[n]*\delta[n] + u[n]*\delta[n-1] + u[n]*\delta[n-2])$$
$$= \underline{\frac{1}{3}(u[n]+u[n-1]+u[n-2])} \qquad \left(= \frac{1}{3}\delta[n] + \frac{2}{3}\delta[n-1] + u[n-2]\right)$$

Alt. lösning: $g[n] = (u*h)[n] \Leftrightarrow G[z] = U[z]\cdot H[z] = $ /Tab. 10:3, 10:1 & 10:2/

$$= \underbrace{\frac{z}{z-1}}_{|z|>1} \cdot \frac{1}{3}(1+z^{-1}+z^{-2}) = \frac{1}{3}\left(\frac{z}{z-1} + z^{-1}\cdot\frac{z}{z-1} + z^{-2}\cdot\frac{z}{z-1}\right); \ |z|>1$$

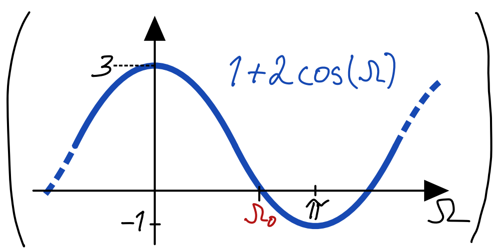
Tab. 10:3 & 9:5 $\Rightarrow$ $\underline{g[n] = \frac{1}{3}(u[n]+u[n-1]+u[n-2])}$

---

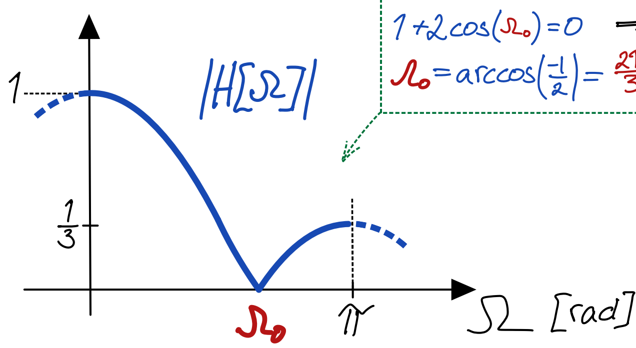
b) $$\underline{H[\Omega]} = \left\{\begin{matrix} \mathcal{F}\{h[n]\} \\ \text{(Tab. 8:1 \& 8:2)} \\ \text{eller} \\ H[z]\big|_{z=e^{j\Omega}} \\ (\text{ok ty } \sum|h[n]|<\infty) \end{matrix}\right\} = \frac{1}{3}(1+e^{-j\Omega}+e^{-j2\Omega}) = \frac{e^{-j\Omega}}{3}(e^{j\Omega}+1+e^{-j\Omega})$$
$$= \frac{e^{-j\Omega}}{3}\left(1+2\cdot\frac{e^{j\Omega}+e^{-j\Omega}}{2}\right) = \underline{\frac{e^{-j\Omega}}{3}(1+2\cos(\Omega))}$$

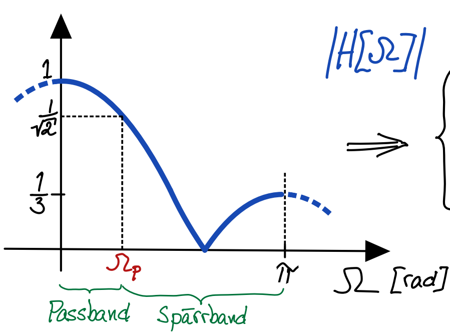
$\Rightarrow$ $\underline{|H[\Omega]| = \frac{1}{3}(1+2\cos(\Omega))}$

$1+2\cos(\Omega_0)=0 \Rightarrow$
$\Omega_0 = \arccos\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$



$\Rightarrow$





$\Rightarrow$ $\left\{\begin{matrix} |H[\Omega]| > \frac{1}{\sqrt{2}}|H[\Omega]|_{max} \text{ för } 0>\Omega>\Omega_p \\ |H[\Omega]| < \frac{1}{\sqrt{2}}|H[\Omega]|_{max} \text{ för } \Omega_p<\Omega<\pi \end{matrix}\right\}$ $\Rightarrow$ Detta är ett <u>LP-filter</u>

---

c) Stabilt system ($\sum_n |h[n]|<\infty$) $\xRightarrow{\text{Formels. sid. 15}}$ För stationär insignal $x[n]=2+5\cos\left(\frac{2\pi}{3}n\right)$ erhålls utsignalen

$$\underline{y[n]} = 2\underbrace{H(0)}_{=1 \text{ enl. graf och } H[\Omega] \text{ ovan}} + 5\cdot\underbrace{\left|H\left[\tfrac{2\pi}{3}\right]\right|}_{=0 \text{ enligt grafen ovan (vid } \Omega_0=\frac{2\pi}{3}\text{ rad)}}\cdot\sin\left(\frac{2\pi}{3}n+\arg H\left[\tfrac{2\pi}{3}\right]\right) = \underline{2}$$

(5) a) LTI-system $\Rightarrow$ Det kaskadkopplade systemet har impulssvar $h[n]=(h_1 * h_2)[n] \Longleftrightarrow$ systemfunktion $H[z]=H_1[z]H_2[z]$ där $h_k[n]$ $(H_k[z])$ är impulssvar (systemfunktion) för system $\mathcal{H}_k$.

$\mathcal{H}_1$: $\mathcal{Z}_{II}\{v[n]-0{,}8v[n-1]\} = \mathcal{Z}_{II}\{3x[n]\} \overset{\text{Tab. 9:4}}{\Longrightarrow} (1-0{,}8z^{-1})V[z] = 3X[z]$

$$\Rightarrow H_1[z] = \frac{V[z]}{X[z]} = \frac{3}{1-0{,}8z^{-1}} = \frac{3z}{z-0{,}8} \; ; \; |z|>0{,}8$$

(Ty kausalt system enl. uppgift)

(Vid systemanalys betraktas alltid energifritt system $\Rightarrow V[z]=V_{zs}[z]$)

$\mathcal{H}_2$: $h_2[n]=0{,}2^n \cdot u[n] \overset{\text{Tab. 10:4}}{\Longrightarrow} H_2[z] = \frac{z}{z-0{,}2} \; ; \; |z|>0{,}2$

$$\Rightarrow H[z] = \frac{3z}{z-0{,}8}\cdot\frac{z}{z-0{,}2}$$

(Enklare att P.B.U. $\frac{H[z]}{z}$ än $H[z]$)

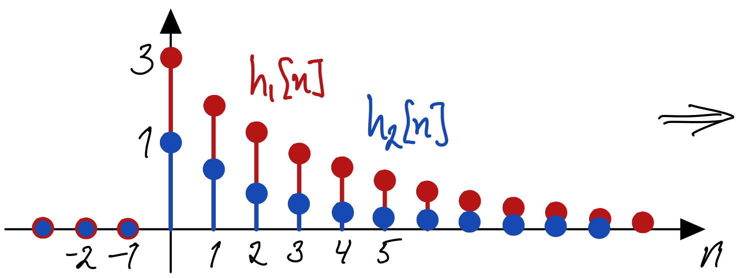
$$\Rightarrow \frac{H[z]}{z} = \frac{3z}{(z-0{,}8)(z-0{,}2)} = /\text{P.B.U.}/ = \frac{4}{z-0{,}8} - \frac{1}{z-0{,}2}$$

$$\Rightarrow H[z] = \underset{(|z|>0{,}8)}{\frac{4z}{z-0{,}8}} - \underset{(|z|>0{,}2)}{\frac{z}{z-0{,}2}} \overset{\text{Tab. 10:4}}{\Longrightarrow} h[n] = (4\cdot 0{,}8^n - 0{,}2^n)u[n]$$

Alternativ lösning m.h.a. **faltning**: $h[n]=(h_1*h_2)[n] = \sum_{m=-\infty}^{\infty} h_1[m]h_2[n-m]$

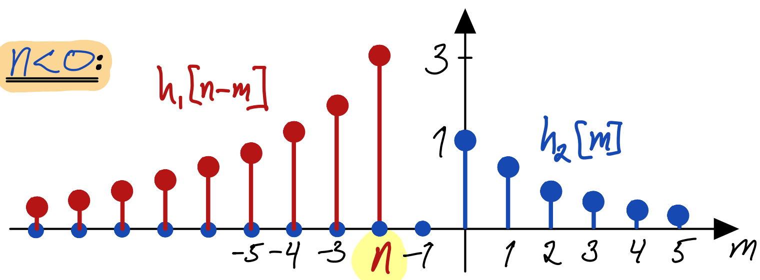
(eller $\sum_{m=-\infty}^{\infty} h_1[n-m]h_2[m]$)

där $h_1[n] = \mathcal{Z}^{-1}\{H_1[z]\} \overset{\text{Tab. 10:4}}{=} 3\cdot 0{,}8^n u[n]$ & $h_2[n] = 0{,}2^n u[n]$



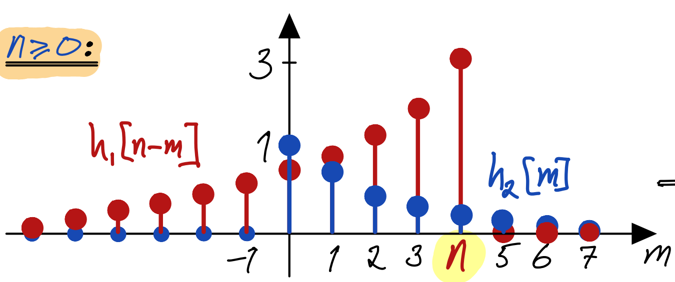
$\Longrightarrow$

$n<0$:



$\Rightarrow h_1[n-m]h_2[m] = 0 \;\; \forall m \Rightarrow h[n]=0$

$n\geq 0$:

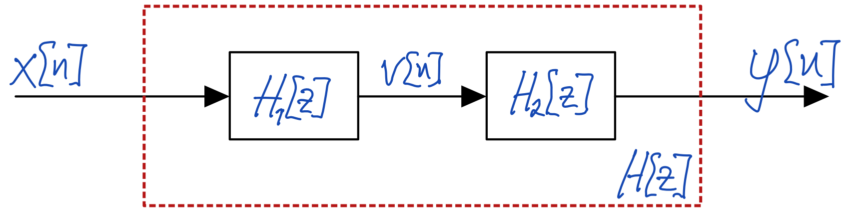


$$\Rightarrow h[n] = \sum_{m=0}^{n} 3\cdot 0{,}8^n \cdot 0{,}2^n = 3\cdot 0{,}8^n \sum_{m=0}^{n}\left(\frac{0{,}2}{0{,}8}\right)^n$$

$$= 3\cdot 0{,}8^n \cdot \frac{1-\left(\frac{1}{4}\right)^{n+1}}{1-\frac{1}{4}} = 4(0{,}8^n - 0{,}25\cdot 0{,}2^n)$$

$$\Rightarrow h[n] = (4\cdot 0{,}8^n - 0{,}2^n)u[n]$$

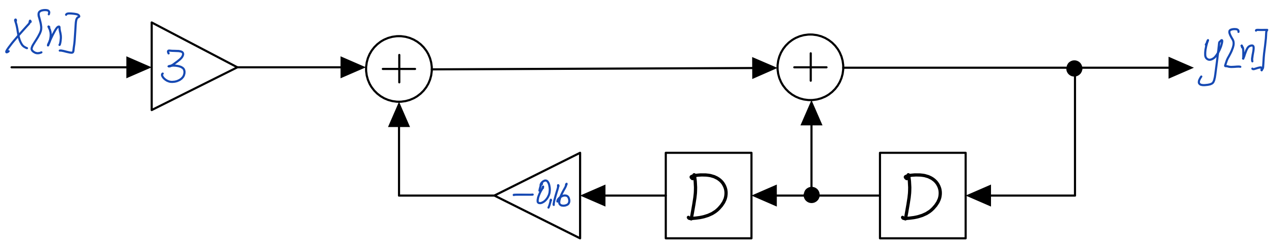
b)



$$H[z] = \frac{3z^2}{(z-0{,}8)(z-0{,}2)} = \frac{3z^2}{z^2 - z + 0{,}16} = \frac{3}{1 - z^{-1} + 0{,}16z^{-2}} = \frac{Y[z]}{X[z]}$$

$$\Rightarrow Y[z] - z^{-1}Y[z] + 0{,}16z^{-2}Y[z] = 3X[z] \overset{\text{Tab. 9:4}}{\Longrightarrow} y[n] - y[n-1] + 0{,}16y[n-2] = 3x[n]$$

$$\Rightarrow \underline{y[n] = 3x[n] + y[n-1] - 0{,}16y[n-2]} \quad \Rightarrow$$



Alternativt kan de två delsystemen realiseras var för sig:

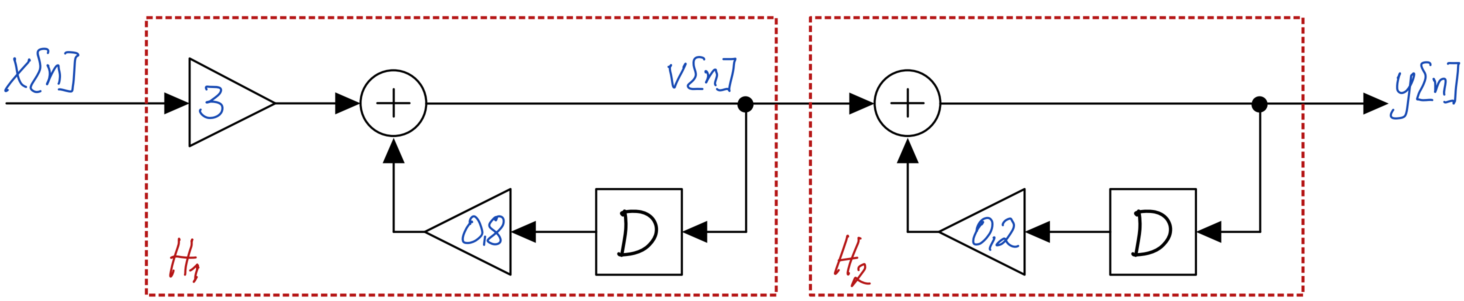
H_1: Differensekvationen ger $\underline{v[n] = 3x[n] + 0{,}8v[n-1]}$ (I)

H_2: $H_2[z] = \dfrac{z}{z-0{,}2} = \dfrac{1}{1-0{,}2z^{-1}} = \dfrac{Y[z]}{V[z]}$

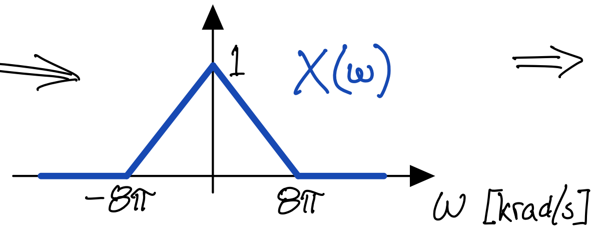
$$\Rightarrow Y[z] - 0{,}2z^{-1}Y[z] = V[z] \overset{\text{Tab. 9:4}}{\Longrightarrow} y[n] - 0{,}2y[n-1] = v[n]$$

$$\Rightarrow \underline{y[n] = v[n] + 0{,}2y[n-1]} \quad \text{(II)}$$

(I) & (II) $\Rightarrow$



6 a) $\omega = 2\pi f$ $\Rightarrow$



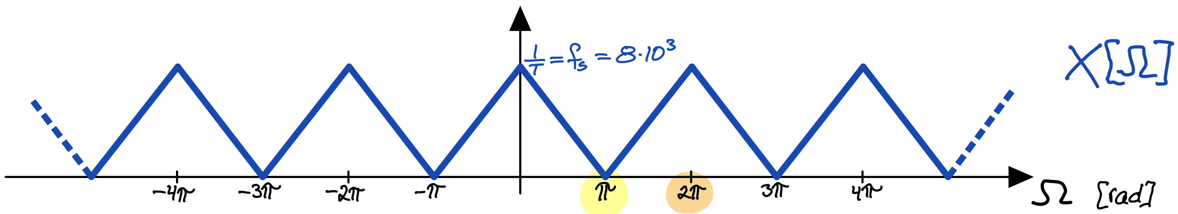
$\Rightarrow$ Bandbredd $W = 8\pi$ krad/s

$x[n] = x(nT)$ $\Rightarrow$ Poissons summationsformel (formels. sid. 9):

$$\left(\mathcal{F}\{x[n]\} = \right) \quad X[\Omega] = \frac{1}{T}\sum_{n=-\infty}^{\infty} \underbrace{X\left(\frac{\Omega - n2\pi}{T}\right)}_{= X(\omega)\big|_{\omega = \frac{\Omega - n2\pi}{T}}}$$

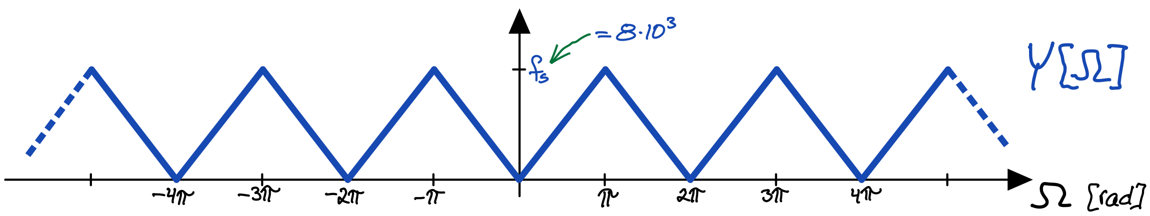
$f_s = 8$ kHz $\Rightarrow$ $\omega_s = 2\pi f_s = 16$ krad/s $= 2W$

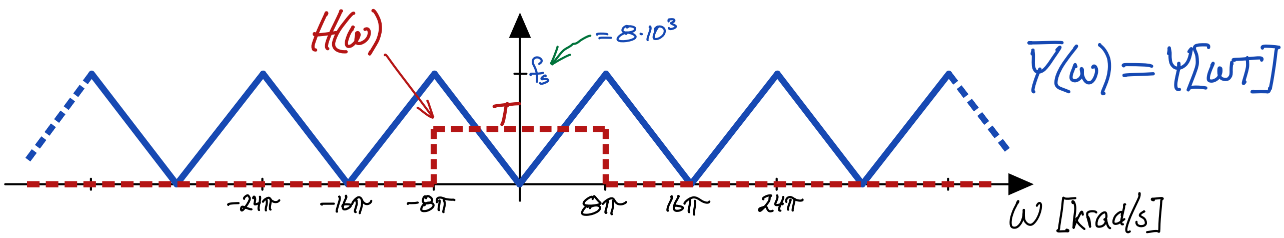
$\Rightarrow$ Ingen vikning, samplingsteoremet är precis uppfyllt:



(Sampling $\Rightarrow$ $\omega \rightarrow \Omega = \omega T$ $\Rightarrow$ $\omega_s \rightarrow \omega_s T = 2\pi$ rad och $W = \frac{\omega_s}{2} \rightarrow \pi$ rad)

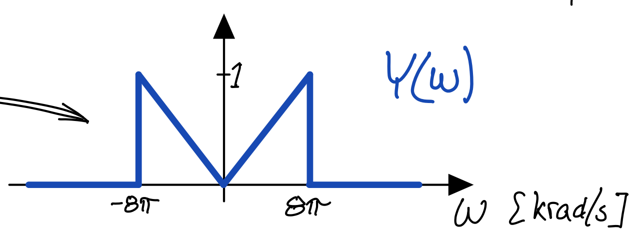
$$y[n] = (-1)^n x[n] \xrightarrow{\text{Tab 7:7}} Y[\Omega] = X[\Omega - \pi]$$





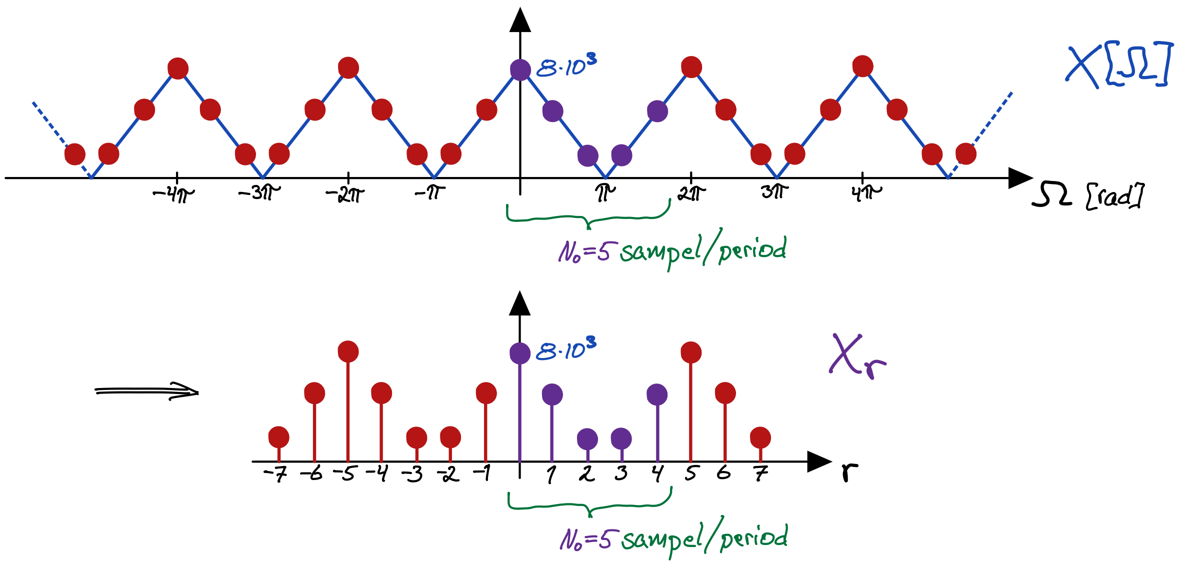
Rekonstruktion Ideal rekonstruktion, samma $f_s = \frac{1}{T}$ som vid samplingen:

$Y(\omega) = \overline{Y}(\omega) \cdot H(\omega)$ $\Rightarrow$



b) $X_r = X[\Omega]\big|_{\Omega = r\cdot\frac{2\pi}{N_0}}$, dvs.

X_r är en sampling av $X[\Omega]$ i $N_0 = 5$ punkter/period:



(X_r är även en sampling av $X[z]$ i N_0 punkter längs enhetscirkeln:

$$X_r = X[z]\big|_{z=e^{jr\Omega_0}}, \text{ där } \Omega_0 = \frac{2\pi}{N_0}$$
)